THE SIMPLE CHAOTIC ENDOGENOUS GROWTH MODEL: U.S.

Vesna JABLANOVIC*

Abstract. *The AK model of economic growth is an endogenous growth model. The key property of AK endogenous growth model is the absence of diminishing returns to capital. The simplest version of endogenous model is AK models which assume constant exogenous saving rate and fixed level of technology. The basic aims of this paper are: firstly, to provide a relatively simple chaotic AK growth model that is capable of generating stable equilibria, cycles, or chaos, and secondly, to analyze the local GDP growth stability in U.S. in the period 1990-2016. This paper confirms stable economic growth in U.S. in the observed period.*

Keywords: *Economic Growth, AK Model, Stability, Chaos.*

1. Introduction

In the late 1980s, the endogenous growth theory emerged. This theory introduced some important inputs such as knowledge capital and learning by doing, which could generate increasing returns.

The basic idea of Romer (1986) approach is that technology grows in proportion to the macroeconomic capital stock, potentially offsetting the effects of diminishing returns. Capital in such a setting should be considered as a broad concept, including human and intangible capital. This approach is currently known as the ″AK approach″ because it results in a production function of the form $Y=AK$ with A constant. The essential idea of the Romer (1986) model is that knowledge can be considered as a kind of renewable capital good (K).

According to the endogenous growth literature, permanent changes in variables that are affected by government policy lead to permanent changes in growth rates. This is the result in both the early *AK* growth models of Romer (1986, 1987), Lucas (1988), and Rebelo (1991), as well as in subsequent models focusing more explicitly on endogenous techno-logical change by Romer (1990), Grossman and Helpman (199la, 1991b) and Aghion and Howitt (1992).

* University of Belgrade, Faculty of Agriculture, Serbia, **vesnajab@ptt.rs**

The basic aims of this paper are: firstly, to provide a relatively simple chaotic AK growth model that is capable of generating stable equilibria, cycles, or chaos [Jablanovic (2011, 2013)], and secondly, to analyze the local GDP growth stability in U.S. in the period 1990-2016.

2. The model

The chaotic *AK* growth model is presented by the following equations:

$$Y_t = A K_t \qquad A > 0 \tag{1}$$

$$Y_t = C_t + I_t + G_t \tag{2}$$

$$G_t = \alpha Y_t^{\,2} \tag{3}$$

$$C_t = \beta Y_t + \gamma K_t \tag{4}$$

$$K_{t+1} = I_t + (1-\delta) K_t \tag{5}$$

with Y – **re**al output, K – the stock of capital, I – investment, C – consumption, G – government expenditure, A – some positive constant that reflects the level of the technology, δ – rate of depreciation, α – the government expenditure rate, β – the average propensity to consume, γ – the coefficient which explains relation between the stock of capital and consumption.

(1) shows the *AK* model production function *K* embodies both physical capital and human capital. The parameter A stands for the level of technology, and is assumed constant. Production depends only on the reproducible factor (K). There are no diminishing returns to this factor. (2) shows GDP (Y) as the sum of consumption (C), investment (I), government spending (G). (3) shows the government spending function. (4) assumes that consumption (C) is a linear function of income (Y), and wealth (capital). Equation (5) shows how the capital stock changes over time. Here δ is the rate of physical depreciation so that between year t and year $t+1$, δK_t units of capital are lost from depreciation. But during year t, there is investment (I_t) that yields new capital in the following year.

Now, putting (1), (2), (3), (4), and (5) together we immediately get:

$$Y_{t+1} = [A(1-\beta) + \gamma + (1-\delta)]\, Y_t - \alpha\, A Y_t^2 . \tag{6}$$

Further, it is assumed that the current value of the real output is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the real output growth rate depends on the current size of the real output, *Y*, relative to its maximal size in its time series Y^m. We introduce *y* as $y = Y / Y^m$. Thus y range between 0 and 1. Again we index *y* by *t*, i.e., write y_t to refer to the size at time steps $t = 0,1,2,3,...$ Now growth rate of the real output is measured as

$$y_{t+1} = [A(1-\beta) + \gamma + (1-\delta)]\, y_t - \alpha\, A y_t^{\,2} \tag{7}$$

This model given by equation (7) is called the logistic model. For most choices of *A*, α, β, γ, and δ there is no explicit solution for (7). Namely, knowing *A*, α, β, γ, and δ measuring y_0 would not suffice to predict y_t for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect – the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

This kind of difference equation (7) can lead to very interesting dynamic behavior, such as cycles that repeat themselves every two or more periods, and even chaos, in which there is no apparent regularity in the behavior of y_t. This difference equation (7) will possess a chaotic region. Two properties of the chaotic solution are important: firstly, given a starting point y_0 the solution is highly sensitive to variations of the parameters A, α, β, γ, and δ; secondly, given the parameters A, α, β, γ, and δ, the solution is highly sensitive to variations of the initial point y_0. In both cases the two solutions are for the first few periods rather close to each other, but later on they behave in a chaotic manner.

3. The logistic equation

It is possible to show that iteration process for the logistic equation

$$z\,t+1 = \pi\, z\, t\,(1 - z\, t), \quad \pi \in [0, 4], \quad zt \in [0, 1] \tag{8}$$

is equivalent to the iteration of growth model (7) when we use the identification

$$z_t = \frac{\alpha\, A}{[A(1-\beta) + \gamma + (1-\delta)]}\, y_t \ \text{ and } \ \pi = [A(1-\beta) + \gamma + (1-\delta)] \tag{9}$$

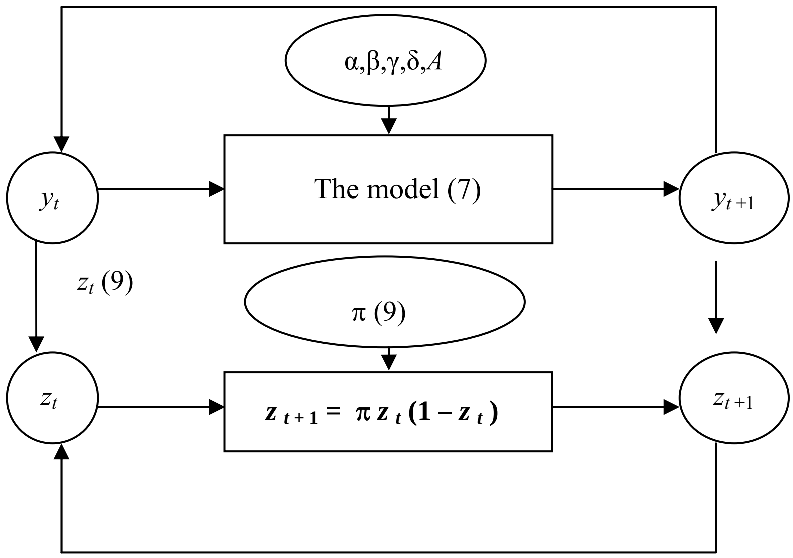

Figure 1. Two quadratic iteratiors running in phase are tightly coupled by the transformations indicated.

Using (7) and (9) we obtain:

$$z_{t+1} = \frac{\alpha\ A}{[A(1-\beta)+\gamma+(1-\delta)]}\ y_{t+1}$$

$$= \frac{\alpha\ A}{[A(1-\beta)+\gamma+(1-\delta)]}\ \{[A(1-\beta)+\gamma+(1-\delta)]y_t - \alpha\ A$$

$$= \alpha\ A\ y_t - \frac{\alpha^2\ A^2}{[A(1-\beta)+\gamma+(1-\delta)]}\ y_t^{\ 2}.$$

On the other hand, using (8) and (9) we obtain:

$$z_{t+1} = \ \pi\ z_t\ (\ 1 - z_t\ ) =$$

$$[A(1-\beta)+\gamma+(1-\delta)]\frac{\alpha\ A}{[A(1-\beta)+\gamma+(1-\delta)]}\ y_t\left\{1-\frac{\alpha\ A}{[A(1-\beta)+\gamma+(1-\delta)]}\ y_t\right\} =$$

$$= \alpha\ A\ y_t - \frac{\alpha^2\ A^2}{[A(1-\beta)+\gamma+(1-\delta)]}\ y_t^{\ 2}.$$

Thus we have that iterating (7) is really the same as iterating (8) using (9). It is important because the dynamic properties of the logistic equation (8) have been widely analyzed (Li and Yorke (1975), May (1976)).

It is obtained that:

(i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$;

(ii) For $1 < \pi < 3{,}57$ there exist fixed points the number of which depends on π;

(iii) For $1 < \pi < 2$ all solutions monotnically increase to $z = (\pi - 1) / \pi$;

(iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1) / \pi$;

(v) For $3 < \pi < 4$ all solutions will continously fluctuate;

(vi) For $3{,}57 < \pi < 4$ the solution become "chaotic" wihch means that there exist totally aperiodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever.

4. The Empirical Evidence

The main aim of this paper is to analyze the economic growth stability in the period 1990-2016, in U.S. (www.imf.org) by using the presented non-linear, logistic growth model (7):

$$y_{t+1} = \pi \, y_t - \upsilon \, y_t^{\,2} \qquad (7)$$

where y – the real output, $\pi = [A\,(1 - \beta - \gamma) + (1 - \delta)]$, $\upsilon = A\alpha$.

Firstly, data on the gross domestic product are transformed (www.imf.org) from 0 to 1, according to our supposition that actual value of the gross domestic product, Y, is restricted by its highest value in the time-series, Y^m. Further, we obtain time-series of $y = Y / Y^m$. Now, the model (7) is estimated. The results are presented in Table 1.

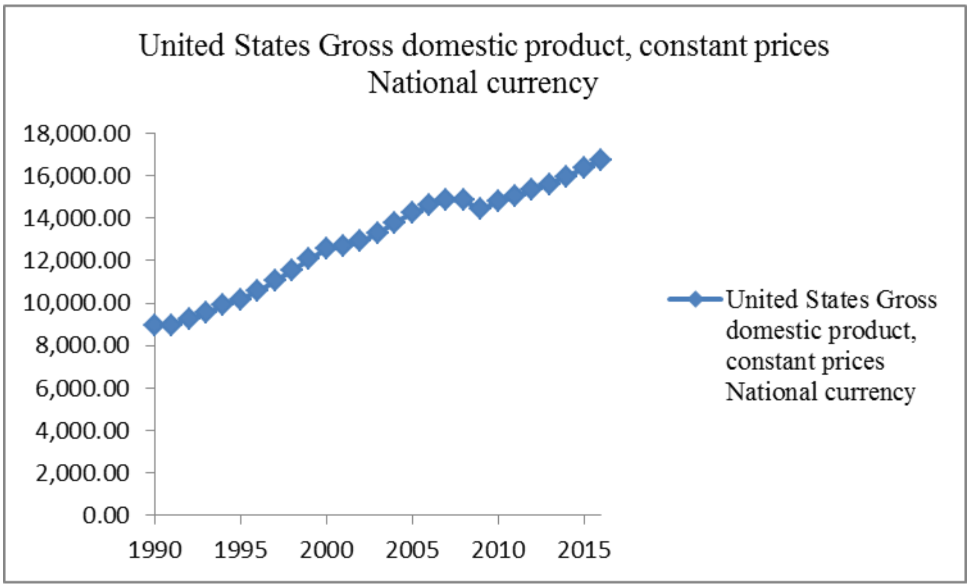

Figure 2. Gross domestic product (constant prices, national currency) in the period 1990-2016: U.S. (www.imf.org).

Table 1

The estimated model (7): U.S., 1990-2016 (www.imf.org).

U.S. :	R =.99640	Variance explained: 99.281%	
		π	υ
	Estimate	1.06330	.050415
	Std.Err.	.02029	.024659
	t(24)	52.40208	2.044486
	p-level	.00000	.052024

5. Conclusion

This paper creates the simple chaotic *AK* growth model. The model (7) has to rely on specified parameters A, α, β, γ, and δ and initial value of the real output y_0. But even slight deviations from the values of parameters: A, α, β, γ, δ, and initial value of the real output, y_0 show the difficulty of predicting a long-term economic behavior.

A key hypothesis of this work is based on the idea that the coefficient $\pi = [A(1-\beta)+\gamma+(1-\delta)]$ plays a crucial role in explaining local stability of the real output, where, δ is the constant depreciation rate, α – the government expenditure rate, β – the average propensity to consume, γ – the coefficient which explains relation between the stock of capital and consumption.

An estimated value of the coefficient π (1.06330) confirms stable economic growth in U.S. in the observed period.

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