ELASTIC COLLISIONS OF HARD SPHERES VERSUS WEALTH EXCHANGE INTERACTIONS

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Abstract. *We discuss the widely accepted deeper analogy between the theory of market economics and the kinetic theory of ideal gases. We first derive the microscopic collision equations of a simplified model of a gas of hard spheres of arbitrary dimension. Assuming that the distribution of the velocities is identical in all regions of space we avoid tracking the spatial position of the individual particles during our random walk. Instead, we simply select randomly pairs of particles which are supposed to collide and to exchange a fraction of their energies. In spite of this dramatic simplification, commonly adapted in wealth exchange models, the velocity as well as the resulting energy distributions compare well with the theoretical Maxwell-Boltzmann distributions.*

The microscopic collision equations which describe the time evolution of the macroscopic variables are then compared with those commonly adapted in various wealth exchange models. It turns out that their formal structure is identical, however the dynamical outcome depends strongly on the choice of the microscopic interaction rules which are often taken "ad hoc" in these models. Depending on the choice of the microscopic interactions, which might not necessarily reflect the physics of collision, one can find all kinds of distributions such as uniform, Gaussian, Gamma, inverse power law and to some extent distributions which do not even manifest a stable equilibrium. However, as long as the energy of the two agents mixes and is randomly distributed between the two agents we recover the familiar Maxwell-Boltzmann distribution, where the theory of molecular chaos and the central limit theorem play a crucial role.

1. Introduction

The Maxwell Boltzmann velocity distribution applies to ideal gases, where the particles do not constantly interact with each other but move freely between short collisions, where the only interactions taken into account are binary collisions. It describes the probability of a particle's velocity, its momentum or energy as a function of the mass and the

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temperature. Provided that the probability distribution is assumed to be independent of the position of the particles, we have the homogeneous Boltzmann equation:

$$\frac{\partial P}{\partial t} = Q(P(t,\vec{v},\vec{v}')) \tag{1}$$

which can be regarded as a cornerstone of statistical mechanics [11]. Here $\vec{v}$ and $\vec{v}$ specify the velocity vectors of a particle before and after the collision, respectively. The collision operator Q accounts for all kinds of binary collisions which preserve the energy and the momentum. It forms the basis of the kinetic theory of gases, which explains many fundamental gas properties such as velocity and energy probability distributions. To be more specific, the Boltzmann equation describes the temporal evolution of the probability distribution of the density of a cloud of particles. The partial derivative of the distribution function $P(t,\vec{v},\vec{v}')$ on the left-hand side in Eq. (1) represents the explicit time variation of the distribution function, while the right-hand side of the equation represents the effect of collisions. In the special case of short-range interactions in the hard sphere model, where particles are spheres only interacting by contact, one assumes for the post-collision velocities [11]:

$$\vec{v}_1' = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) + \frac{1}{2}\mathbf{R}(\vec{v}_1 - \vec{v}_2) \qquad \vec{v}_2' = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) - \frac{1}{2}\mathbf{R}(\vec{v}_1 - \vec{v}_2). \tag{2}$$

Here, the first term is the velocity of the centre of mass, while the difference term, the relative velocity of the two particles, undergoes a rotation via the rotation matrix **R**. The velocity, the energy and all the other macroscopic quantities will have different probability distribution functions, although all these distributions are related. The velocity distribution $P(\vec{v})$ in n-dimensional space is a product whose n factors are independent and normally distributed with zero mean.

We stress already at this point that there are fundamental differences between the collision mechanism of classical gas particles and the wealth exchange interactions of agents, where savings and highly selective random effects play a significant role, while for the classical gas molecular chaos plays the decisive role [3, 11]. In fact, the crucial problem in modelling wealth exchange is the search for meaningful real-life macroscopic rules, which describe how wealth is exchanged in economic trades. In most models currently practiced, these rules are derived from plausible assumptions in an ad hoc manner, which is clearly in marked contrast to Boltzmann's original theory, where the microscopic collisions are governed by strict physical laws given for example by Eq. (2) in the case of the hard sphere problem. On the other hand, wealth exchange models often aim at a Pareto tail of the probability distribution of wealth [8] which is a manifestation of the existence of very rich agents, a manifestation of an unequal distribution of wealth.

2. Elastic collision in arbitrary dimensions

We assume collisions to be elastic, i.e., the total kinetic energy and momentum of the particles are conserved. Let us first treat particles which are all equal and indistinguishable with masses $m_1 = m_2 = 1$. The collision process can then be analyzed in two reference frames: the laboratory frame and centre of mass frame. Assuming that the two particles move with velocities $\vec{v}_1$ and $\vec{v}_2$, the centre of mass of the system moves with the velocity:

$$\vec{v}_s = \frac{1}{2}(\vec{v}_1 + \vec{v}_2). \tag{3}$$

The corresponding velocities in the centre of mass coordinate system are:

$$\vec{v}_{1s} = \vec{v}_1 - \vec{v}_s = \frac{1}{2}\vec{v}_{12} \qquad \vec{v}_{2s} = \vec{v}_2 - \vec{v}_s = -\frac{1}{2}\vec{v}_{12} \tag{4}$$

with $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$. The momentum, identical with the velocities in the equal mass case, are equal in magnitude and opposite in direction. Accordingly, in the centre of mass frame a collision induces a pure rotation of the velocity vectors:

$$\vec{v}'_{1s} = \mathbf{R}\vec{v}_{1s} \qquad \vec{v}'_{2s} = -\mathbf{R}\vec{v}_{2s}. \tag{5}$$

Here **R** specifies a rotation matrix with the axis of rotation and the rotation angle still to be specified. Note that for dimensions $D > 2$ the velocity vectors after the collision do not necessarily lie in the same plane as the velocity vectors before the collision. Back transforming from the centre of mass frame into the laboratory frame the post-collision velocities are:

$$\vec{v}'_1 = \vec{v}_1 + \vec{\Delta} \qquad \vec{v}'_2 = \vec{v}_2 - \vec{\Delta} \tag{6}$$

with the transferred momentum:

$$\vec{\Delta} = \frac{1}{2}(\mathbf{R}\vec{v}_{12} - \vec{v}_{12}) \tag{7}$$

The moments are conserved and the squared velocities take the form:

$$\upsilon_1'^2 = \vec{\upsilon}_1^2 + 2\vec{\upsilon}_1\vec{\Delta} + \vec{\Delta}^2 \qquad \upsilon_2'^2 = \vec{\upsilon}_2^2 - 2\vec{\upsilon}_2\vec{\Delta} + \vec{\Delta}^2 \tag{8}$$

while energy conservation demands:

$$(\vec{\upsilon}_1 - \vec{\upsilon}_2)\vec{\Delta} + \vec{\Delta}^2 = 0. \tag{9}$$

Introducing the cosines of the angles φ1 and φ2 between the transferred momentum $\vec{\Delta}$ and the velocities $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ by:

$$\cos\varphi_1 = \frac{\vec{\upsilon}_1\vec{\Delta}}{|\vec{\upsilon}_1||\vec{\Delta}|} \qquad \cos\varphi_2 = \frac{\vec{\upsilon}_2\vec{\Delta}}{|\vec{v}_2||\vec{\Delta}|} \tag{10}$$

into Eq.(9), we can solve for the magnitude of the transfer vector $|\vec{\Delta}|$ and find

$$|\vec{\Delta}| = |\vec{\upsilon}_2|\cos\varphi_2 - |\vec{\upsilon}_1|\cos\varphi_1. \tag{11}$$

Eventually we have for the post-collision energies:

$$E_1' = E_1 \sin^2\varphi_1 + E_2\cos^2\varphi_2 \qquad E_2' = E_2\sin^2\varphi_2 + E_1\cos^2\varphi_1. \tag{12}$$

Note that in the case of different masses $(m_1 \neq m_2)$ the formalism is as straightforward, however lengthy, and the right hand side of Eq. (12) is not linear in the energy terms E_1 and E_2 any more.

3. Collision geometry for hard spheres

Provided we know the two velocity vectors $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ before the collision, we also know the scattering angle $\varphi_{12} = \sphericalangle(\vec{\upsilon}_1, \vec{\upsilon}_2)$. The angle of rotation φ then allows the calculation of all relevant quantities by means of a simple geometrical problem in two dimensions, where the rotation angle φ can take any value between 0 and π. We eventually find for the angles φ_1 and φ_2:

$$\cos\varphi_1 = \frac{+|\vec{\upsilon}_2|(\cos\varphi_{12} - \cos(\varphi_{12} \pm \varphi)) - |\vec{\upsilon}_1|(1-\cos\varphi)}{\sqrt{2(1-\cos\varphi)}\,|\vec{\upsilon}_{12}|} \tag{13}$$

$$\cos\varphi_2 = \frac{-|\vec{\upsilon}_1|(\cos\varphi_{12} - \cos(\varphi_{12} \mp \varphi)) + |\vec{\upsilon}_2|(1-\cos\varphi)}{\sqrt{2(1-\cos\varphi)}\,|\vec{\upsilon}_{12}|}. \tag{14}$$

Figure 1 depicts a scattering diagram which allows to construct the post-collision velocity vectors Eq. (6) from a pure geometrical aspect. The figure:

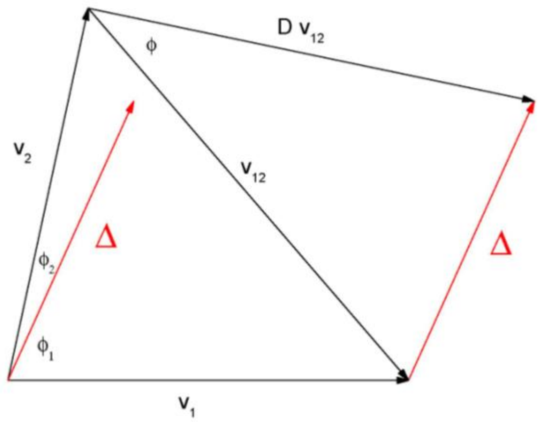

Figure 1. Scattering diagram with the main ingredients, the pre-collision velocity vectors $\vec{v}_1,\vec{v}_2$ which determine the crucial transfer vector $\vec{\Delta}$ order to construct the post-collision: velocities $\vec{v}_1'$ and $\vec{v}_2'$.

also shows that the two angles φ_1 and φ_2 are not independent. They are related to the collision angle $\varphi_{12} = \measuredangle(\vec{v}_1,\vec{v}_2)$ via:

$$|\varphi_1 \pm \varphi_2| = \varphi_{12} \tag{15}$$

where the angles in Eq. (15) are considered to be taken mod (2π). The sign within the absolute values depends on the geometrical situation. Provided that the transfer vector $\vec{\Delta}$ lies within the parallelogram spanned by the two velocity vectors $\vec{v}_1$ and $\vec{v}_2$, we have the plus sign, otherwise we have to take the minus sign. The two angles φ_1 and φ_2 depend uniquely on the pre-collision velocity vectors as well as on the angle of rotation φ. For the hard sphere model the angle of rotation φ is deterministic and depends on the relative space coordinate $\vec{r}_{12}$ of the two particles at the moment of the collision as well on the radii of the particles in terms of the impact parameter [7]. For the ideal gas we make use of the molecular chaos assumption which is part of the kinetic theory of gases. The postulate says that during a two-body collision between particles, there is no correlation of velocity. However, it has been impossible to prove this assumption rigorously, although it is widely thought to be true. Accordingly, the angle of rotation φ can be taken at random from a uniform distribution in the interval $\varphi \in [0:\pi]$. As shown

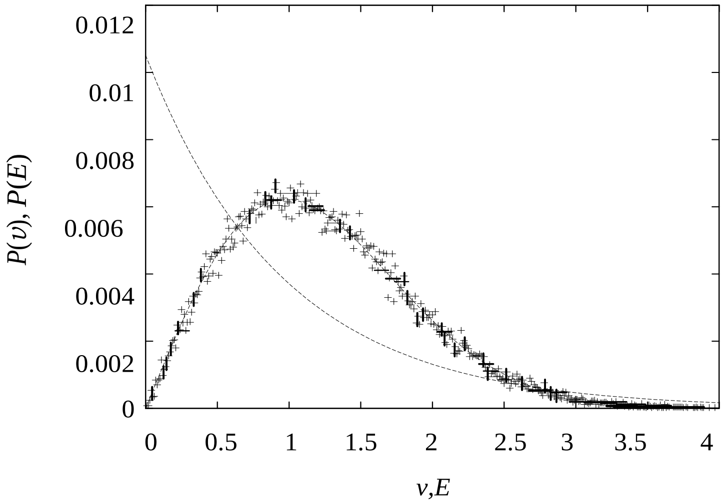

Figure 2. Velocity and Energy distribution P (v) and P (E) according to Eqs. (16) (17) in two dimensions for arbitrary rotation angle φ.

in Figure 2 the velocity distribution is Gaussian:

$$P(\upsilon) \propto \upsilon^{D-1} e^{-\frac{m\upsilon^2}{2kT}} \tag{16}$$

and depends on the space dimension D as well as on the temperature T, where the Boltzmann factor k has been set to one. The probability density function of the energy can directly be calculated by familiar transformation techniques $\upsilon \to E = \frac{m}{2}\upsilon^2$ and we have:

$$P(E) \propto E^{\frac{D}{2}-1} \exp\left(-\frac{E}{T}\right). \tag{17}$$

The energy distribution is the familiar Gamma distribution, where for dimension D = 2 we recover the exponential distribution. Note that there are a variety of ways to derive the Maxwell-Boltzmann statistics.

4. Interacting agents

It is meanwhile widely accepted that in certain aspects a wealth exchange model composed of an assembly of N indistinguishable agents, each of which has a certain wealth E_i (i = 1, ...N), can be attacked with the tools of the Boltzmann legacy [11, 13] even if the interaction rules are not that physical at all. The gas particles correspond to the agents, the energy of the particles are identified with the wealth of the agent, while the binary collisions correspond to wealth-exchange interactions between the agents. The interactions occur pair wise between randomly chosen agents who exchange wealth or energy according to the following interaction rule:

$$E_i' = pE_i + qE_j \quad E_j' = (1-q)E_j + (1-p)E_i. \tag{18}$$

Note that these equations are formally identical with Eq. (12) which describe the energy transfer of hard spheres with $p = \sin^2 \varphi_1$ and $q = \cos^2 \varphi_2$. However, in contrast to wealth exchange models, in the hard spheres problem the two angles φ_1 and φ_2 are specified by the physics of collisions Eq. (2). According to Eqs. (13) and (14) they depend on the two velocity vectors $\vec{v}_1$ and $\vec{v}_2$ as well as on the angle of rotation φ. They also satisfy the relation $|\varphi_1 \pm \varphi_2| = \varphi_{12}$, although in contrast to the physics of collision, wealth exchange models assume that the corresponding interaction coefficients p and q are independent. In terms of the energy transfer we have:

$$E_i' = E_i - \vec{\Delta} \qquad E_j' = E_j + \vec{\Delta} \tag{19}$$

with the transfer term:

$$\vec{\Delta} = pE_i - (1-q)E_j. \tag{20}$$

Agent i transfers a fraction $1 - p$ of its wealth to agent j, while agent j transfers a fraction q of its wealth to agent i. The interaction coefficients p and q, both taken $\in [0, 1]$ are usually drawn from rather specific probability distributions according to the needs and aims of the specific model. Different wealth exchange models differ in their recipes of specifying the microscopic interaction coefficients p and q.

The decisive question now is: How does the probability density distribution $P(E)$ evolve in time during a large number of interactions? The time dependent distribution will obviously depend on the particular interaction rules, in contrast to nature that chooses the universal Maxwell-Boltzmann distribution in the ideal gas problem extremely rapidly [7]. The simplest wealth exchange model is characterized by a completely random exchange of energy following closely the rules of an ideal gas. One assumes that during each interaction the total energy of the two agents E_1 and E_2 mixes and randomly splits between the two agents with equal probability to gain or to lose energy. This choice corresponds to the choice $p = q$ in Eq.(20), where p is a random number chosen uniformly in the interval [0, 1]. This yields the well-known Maxwell-Boltzmann distribution of the energy for $D = 2$:

$$P(E) \propto \exp\left(-\frac{E}{T}\right) \tag{21}$$

with the equilibrium distribution given by the pure exponential, where the temperature T specifies the average energy.

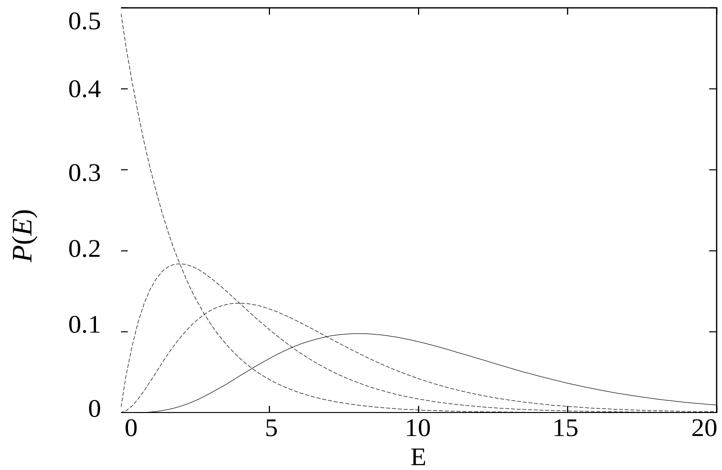

Figure 3. Gamma distribution for T = 2 and different values of k (k = 0.8, 1, 2, and 3.5).

4.1. Saving with fixed fraction of exchange

Features typically incorporated in kinetic trade models are saving effects and a rather specific kind of randomness. Here, saving means that each agent is obliged to keep a certain minimal fraction of his initial wealth which is not involved into the trade. Randomness means that the amount of transferred wealth is not deterministic and depends on a specific probability distribution characteristic for a specific model. It is a generalization of the basic elastic scattering model, where not the full amount of their energy, but only a fraction λ of the energy is involved in the exchange process [2, 12]. The agents save a fraction λ of their energy during the interaction such that the energy part $\lambda(E_1 + E_2)$ mixes and is randomly split between the two agents. The microscopic transfer equations for a quite general model, where a variety of commonly used models stem from, are given by:

$$E_1' = \lambda E_1 + \varepsilon(1-\lambda)(E_1 + E_2) \qquad E_2' = \lambda E_2 - (1-\varepsilon)(1-\lambda)(E_1 + E_2) \quad (22)$$

with the interaction coefficients:

$$p = \lambda + \varepsilon(1-\lambda) \qquad q = \varepsilon(1-\lambda) \quad (23)$$

specified in Eq. (20) The stochastic control parameter $\varepsilon \in [0,1]$, which is chosen at random at each interaction, specifies how the remaining amount for exchange $(1 - \lambda)(E_1 + E_2)$ is shared between the two interacting agents.

This randomly sharing of the "bet" with a random variable ε uniformly distributed in the interval [0, 1] constitutes a rather awkward recipe for professional commercial interactions, while on the other hand, it is much better suited to the ideal gas model. For $\varepsilon = 1$ we recover Angle's [1, 4] one parameter model, the very first description of a wealth exchange model with binary interactions, where the winner or the loser are chosen at random. Here, the loser transfers a fixed fraction λ of his wealth to the winner. Note that winning and losing is an asymmetric process. The loss for the loser is deterministic, since he loses a fraction p of his wealth. In contrast, the gain for the winner is a random process, since he gains a fraction p of the wealth of his randomly chosen partner. Leaving the path of physics, the time evolution of the models seems to relax toward an equilibrium probability density distribution well approximated by the gamma distribution [1]:

$$P_\gamma(E) = \frac{1}{\Gamma(k)} \frac{1}{T} \left(\frac{E}{T} \right)^{k-1} \exp\left(-\frac{E}{T} \right). \tag{24}$$

However, in contrast to the energy distribution of the ideal gas model the fitting parameter k in Eq. (24) which is related to the dimension D, can also take arbitrary positive real values, depending on the control parameter λ, which constitutes an interesting generalization to non-integer dimensions. Only for $p = \frac{1}{2}$, when the loser transfers exactly half of its energy, the solution is analytic with $k = 1$, corresponding to the dimension $D = 2$ and follows the Boltzmann statistics.

4.2. Individual saving with random fraction of exchange

In this model agent i saves an individual fraction λ_i of his energy and invests the fraction $1 - \lambda_i$ of its wealth into the exchange process. In this model the available wealth $(1 - \lambda_i)(E_i + E_j)$ is not equally shared, but in a stochastic way via a randomly chosen control parameter ε. Eventually all agents can become equally rich or poor, i.e. we end up with Gaussian and Gamma like probability distributions, respectively. The microscopic interaction rules are given by:

$$E_i' = E_i + \vec{\Delta} \qquad\qquad E_j' = E_j - \vec{\Delta} \tag{25}$$

with:

$$\Delta = \varepsilon(1 - \lambda_j^\alpha) E_j - (1 - \varepsilon)(1 - \lambda_i^\alpha) E_i. \tag{26}$$

Here the control parameters λ_i (i = 1, ...N) uniformly distributed [0, 1] denote individual savings parameters, which determine the fraction of the energy of the i-th individual is saved. The stochastic control parameter $\varepsilon \in [0, 1]$, which is changed at each interaction, specifies how the remaining

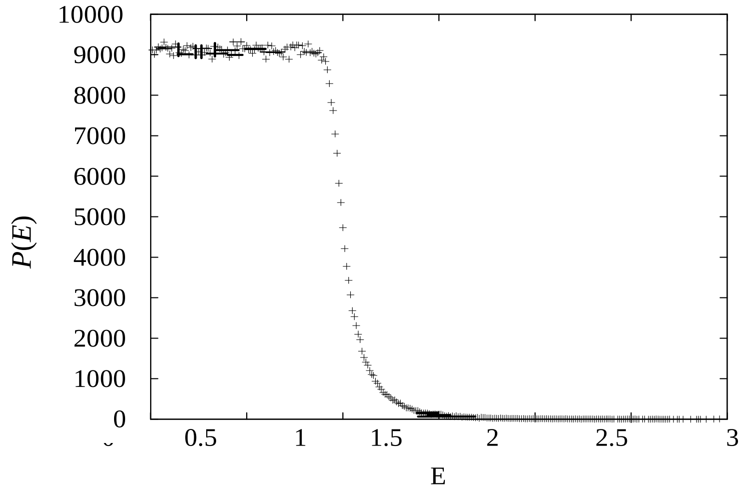

Figure 4. Non-normalized energy distribution for λ_i chosen uniformly and $\alpha >> 1$.

amount for exchange $(1 - \lambda)(E_i + E_j)$ is shared between the two interacting agents. For $\lambda_i = 0$ we recover the unconstrained ideal gas model, where the agents encounter random elastic collisions. The resulting distribution is of the familiar Maxwell-Boltzmann type. In the individual savings model the particles are no longer identical, since the savings parameter λ_i is different for each agent. For different values of the control parameter α which regulates the savings mechanism we find a variety of distributions, even uniform distributions in the low energy part, while the tail is complemented with the famous pareto inverse power law for $\alpha >> 1$ depicted in Figure 4.

4.3. "The rich get richer" model

In this somewhat exotic model always the richer agent wins such that the poorer agent transfers a fraction p of his wealth to the richer agent. This is obviously in marked contrast to the physics of collision, where in general the energetically richer particle transfers a fraction of its energy to the energetically poorer particle, which gives rise to a less unequal distribution. Computer simulations for the "Rich get richer model" reveal that the time evolution of the probability distribution does not reach a stationary state. Since essentially the richer agents become richer, energy conservation implies that with increasing time there will be many more poor agents than rich ones. In the long time limit a small fraction of the agents possesses most of the wealth [6]. Eventually, the whole energy ends up being in the hands of one single agent. This “economic collapse” would correspond to a gas, where one particle has absorbed all the kinetic energy, while the other particles do

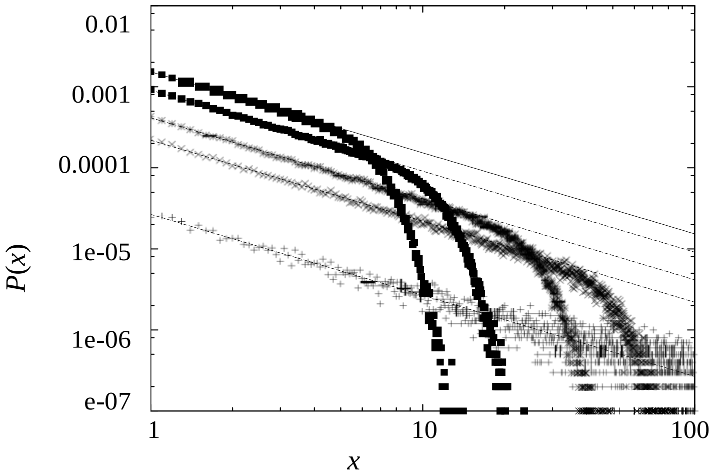

Figure 5. Energy distribution for increasing values of the energy transfer parameter *p* (*p* = 0.2, *p* = 0.5, *p* = 0.6, and *p* = 0.8 from above) for the energetically poorer agent.

not move at all and rather stay at rest. Computer simulations reveal that the low energy part of the energy distribution follows a power law $P(E) \propto \frac{1}{E^k}$ with $k = 1$, while the tail of the distribution seems to follow an exponential law in accord with the Bose-Einstein distribution:

$$P(E) \propto \frac{E^{k-1}}{\exp\left(\frac{E-\mu}{T}\right)-1}. \qquad (27)$$

Note that, provided that the chemical potential μ takes the value zero the Bose-Einstein distribution also an exponentially truncated power law. Moreover, in classical gases the collisions are completely random which results in Maxwell-Boltzmann distributions. In contrast, collisions in bosonic gases are selective, since the particles prefer highly occupied energy states due to the bosonic enhancement in the occupation number the zero momentum natural orbital. We refer to a detailed description and analysis of the “Rich get richer” model in these proceedings [6]. However the appearance of Bose condensates of more than 90% in the energy distribution might be a finite size effect due to the discrete binning procedure such that almost all poor agents will be found in the very first bin.

5. Discussion

Due to substantial differences between the collision dynamics of ideal gas particles and the modelling of wealth exchange interactions, only models based most closely on the kinetic theory of gases predict the familiar Maxwell-Boltzmann distributions. Whenever the interactions are governed by more or less exotic rules which seriously constrain the energy exchange they produce a memory effect on the agents. The result is the appearance of more or less unphysical and exotic energy distributions. By contrast, as long as the rules are chosen randomly in an unconstrained random fashion in analogy to the molecular chaos postulate, the distributions turn out to be physical and obey the familiar Maxwell-Boltzmann statistics. Macroscopically, the richness of the steady states of the energy distributions for wealth exchange models is the essential difference with respect to the theory of Maxwell particles. While the Maxwell-Boltzmann distribution is the universal stationary profile for the distribution of ideal gas particles, those for wealth exchange models can widely vary. They take forms such as uniform distributions, truncated exponential distributions, Gamma distributions, Gaussian distributions, mixed exponential and inverse power law distributions, the celebrated Pareto distributions. Note that most of these piecewise defined distributions, which are not even known analytically, can also be found in the theory of complex biological systems. On the other hand, depending upon the specific choice of the saving mechanism and the stochastic nature of the exchange interactions, the studied systems produce the desired wealth distributions, either of the Boltzmann type or those with the celebrated Pareto law. To conclude, the analogy between the theory of market economics and the kinetic theory of ideal gases does not seem to be that deep and direct as commonly believed. However there is no doubt that Maxwell-Boltzmann theory can serve as the first rate tool for the analytic and numerical analysis of wealth exchange interactions, even if these interactions are far away from the path of physics.

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REFERENCES

[1] J. Angle, *The Inequality Process vs. The Saved Wealth Model. Two Particle Systems of Income Distribution: Which Does Better Empirically*?; online at http://mpra.ub.uni-muenchen.de/20835/.

[2] Arnab Chatterjee and Parongama Sen, *Agent dynamics in kinetic models of wealth exchange*, Phys. Rev. E 82, 056117 (2010).

[3] B. Duering, D. Matthes and G. Toscani, *A Boltzmann-type approach to the formation of wealth distribution curves*, Riv. Mat. Univ. Parma (8)1 (2009), 199-261.

[4] K. E. Kürten and F. V. Kusmartsev, *About distribution of money in a free market economy*, ENEC, V3, 7- 21 (2010).

[5] K. E. Kürten and F. V. Kusmartsev, EPL (93) 28003 (2011).

[6] K. E. Kürten and F. V. Kusmartsev, *When rich get richer there arises a financial crisis and Bose-Einstein condensation in a wild economy* (see this volume) (2011).

[7] J. Novak and A. Bortz, *The evolution of two-dimensional Maxwell-Boltzmann distribution*, American J. of Physics, 38, 12(1970).

[8] V. Pareto, *Cours d'economie politique (Economics course)* (1897).

[9] Marco Patriarca Anirban Chakraborti, and Kimmo Kaski, Phys. Rev. E 70, 016104 (2004).

[10] K. Staliunas, *Bose-Einstein Condensation in Financial Systems*, Nonlinear Analysis: Modelling and Control, 10, 3, 247-256 (2005).

[11] G. Toscani, *Boltzmann Legacy and wealth distribution*, Science and Culture, 8/9, (2010).

[12] Probability distributions in conservative energy exchange models of multiple interacting agents, J. Phys.: Condensed Matter 19(2007) 065138.

[13] V. M. Yakovenko and J. B. Rosser, Jr., *Colloquium: Statistical mechanics of money, wealth, and income*, Reviews of Modern Physics 81, 1703(2009).