MODELING THE REASONS FOR FIRMS' GROWTH: A CONTROL-THEORETIC APPROACH

Matti ESTOLA*

Abstract. *Static neoclassical framework cannot be applied in modeling time dependent processes or increasing returns to scale in production. The dynamization of the neoclassical theory of a firm by dynamic optimization, on the other hand, assumes inconsistent profit functions with the former. As a solution to these problems, we present a dynamic theory of a firm consistent with the static neoclassical one. A firm's growth may originate from increasing demand or decreasing costs with time, or increasing returns to scale in production (Estola 2001). We model the reasons causing increasing returns in a firm's production process and give a block diagram of the model to reveal its control theoretic nature. (JEL B41, D21, D24)*

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1. Introduction

According to Philip Mirowski (1989a), neoclassical economics bases on two distinct elements: egoistic economic agents by Adam Smith (utility maximizing consumers by William Stanley Jevons, Carl Menger and Leon Walras) and the mathematical metaphor of classical mechanics. The latter can be understood by the progenitors of the neoclassical theory who were engineer level physicists. Concept equilibrium in economics was borrowed from physics by Nicolas-Francois Canard at 1801 (Mirowski 1989b). Although equilibrium is a *balance of forces situation*, in economics the balancing forces have not been defined. However, to understand the adjustment process we should define the forces that 'push' economic quantities into their equilibrium states in stable cases, or cause their evolution with time in non-stable cases.

The existence of forces acting upon economic quantities can be argued indirectly; every changing quantity (price, wage, exchange rate etc.) tells the existence of reasons (forces) causing these changes. This is analogous with arguing the existence of the gravitational force field by dropping a pen; without the force field the pen would not move. Franklin

* University of Eastern Finland, Faculty of Social Sciences and Business Studies, P.O. Box 111, FIN-80101 Joensuu, Finland, e-mail: matti.estola@uef.fi.

M. Fisher (1983 pp. 9-12) writes: *"... I now briefly consider the features that a proper theory of disequilibrium adjustment should have ... if we are to show under what conditions the rational behavior of individual agents drives an economy to equilibrium. ... Such a theory must involve dynamics with adjustment to disequilibrium over time modeled. ...the most satisfactory situation would be one in which the equations of motion of the system permitted an explicit solution with the values of all the variables given as specific, known functions of time. ... Unfortunately, such a closed-form solution is far too much to hope for. ...the theory of the household and the firm must be reformulated and extended where necessary to allow agents to perceive that the economy is not in equilibrium and to act on that perception. ... A convergence theory that is to provide a satisfactory underpinning for equilibrium analysis must be a theory in which the adjustments to disequilibrium made by agents are made optimally".*

According to Kenneth Arrow & Frank Hahn (1971, p. 12), the accepted definition for market stability is that of Paul Samuelson (1942) who writes: "*In the history of mechanics, the theory of statics was developed before the dynamical problem was even formulated. But the problem of stability of equilibrium cannot be discussed except with reference to dynamical considerations ... we must first develop a theory of dynamics*". Andreu Mas-Colell et al. (1995, p. 620) write*: "A characteristic feature that distinguishes economics from other scientific fields is that, for us, the equations of equilibrium constitute the center of our discipline. Other sciences, such as physics or even ecology, put comparatively more emphasis on the determination of dynamic laws of change. The reason, informally speaking, is that economists are good (or so we hope) at recognizing a state of equilibrium but are poor at predicting precisely how an economy in disequilibrium will evolve. Certainly there are intuitive dynamic principles: if demand is larger than supply then price will increase, if price is larger than marginal costs then production will expand".*

2. Dynamic Control Theoretic Model

We base our modeling on the principles stated by above mentioned authors. We define a force vector acting upon the use of inputs of a firm, and use this force in modeling the dynamics of production in real time. The possible asymptotic steady-state of a firm is the neoclassical one: marginal revenues equal marginal costs for all inputs, and the equilibrium use of every input maximizes the firm's profit. To define the *economic forces acting upon the use of inputs of a firm,* we assume that the firm's managers like to better the firm's situation if possible, as Fisher stated above. We believe that '*economic agents' desire to better their situation*' is the fundamental cause for the observed dynamics in economies.

Dynamic control theoretic models of firms' behavior completely base on dynamic optimization; see Alain Bensoussan et al. (1974) or Alpha C. Chiang (1992). In physics, however, Hamilton's and Newton's modeling principles yield the same equations of motion, and Newton's method applies differential equations while Hamilton's principle requires dynamic optimization. Due to this relative simplicity, most engineering applications of control theory rely on Newton's principle. In engineering models it is also common to present a block diagram of a system before its behavior is analyzed formally (see S. A. Marshall (1978) or Katsuhiko Ogata (1997)). We propose a similar simplification for control theoretic modeling in economics as Newton's principle offers in physics. The *force vector* that causes the dynamics of use of every input of a firm guides the firm with time into its profit maximizing state, and so dynamic optimization is not needed. The Newtonian type of a *dynamic law for the firm's adjustment of its use of inputs* is used as a building block in our model as is common in engineering (see Fig. 1). By the model we study the reasons for production dynamics at firm level.

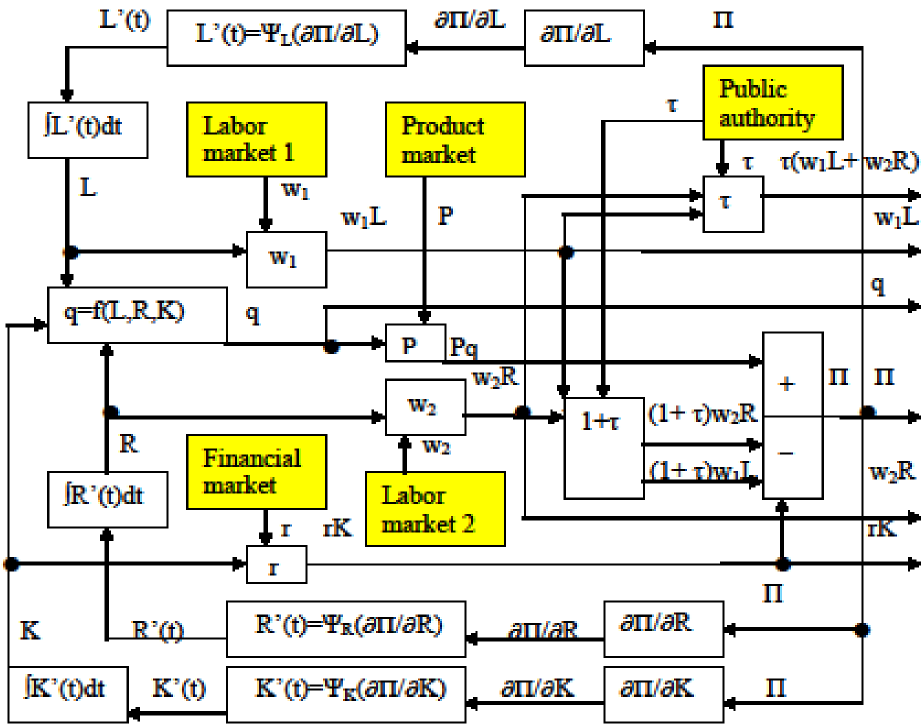

Figure 1. Block Diagram of the Production System of a Firm.

In Figure 1, the ‘flows’ denoted by arrows are named by the ‘flowing’ quantities. Symbol • means that the crossing ‘flows’ are connected, that is, the same flow is in both lines; otherwise crossing lines are assumed not to interact with each other. White rectangles affect the ‘flows’ as is shown, and yellow rectangles of external impulses to the system. For every rectangle, the input and output flows are named. In the single [+/–] – box, the monetary flow entering the upper part is added with a positive sign, and those entering the lower part are added with a negative sign. The output flow from this box is the profit Π. The diagram shows how profit-seeking, the ultimate goal of the firm, controls the production process. All three inputs in the production are adjusted by a closed-loop mechanism where the feed-back bases on the marginal profitability of the input. The real output of the system is q, and the monetary outputs are w_1L and w_2R for the two types of labor, $\tau(w_1L + w_2R)$ for the government, rK for the lenders of foreign capital or investors of own capital, and Π to the firm’s owners. The main definitions are:

$$q = f(L, R, K), \;\; \Pi = Pq - (1+\tau)w_1L - (1+\tau)w_2R - rK,$$

$$\partial\Pi / \partial L = P(\partial f / \partial L) - (1+\tau)w_1; \quad \partial\Pi / \partial R = P(\partial f / \partial R) - (1+\tau)w_2,$$

$$\partial\Pi / \partial K = P(\partial f / \partial K) - r, \;\; x'(t) = \Psi_x(\partial\Pi / \partial x), \;\; x = L, R, K.$$

The theory of endogenous economic growth by Paul Romer, Robert Lucas etc. rose as a critique to Robert Solow’s (1956) model. Romer (1990) explains macro-level economic growth by firms' investments in *human capital,* and Lucas (1988) measures an individual's human capital by his *skills*. Although Romer and Lucas both claim that the development of human capital is the *unobservable magnitude or force behind economic growth,* by human capital they mean different things. The reasons that increase the efficiency of labor may be exogenous or endogenous for a firm. Exogenous reasons are *learning by doing at work* – that takes place inside firms but does not require an investment in it – and better education or technological level in the society. Endogenous reasons are an increase in the number of capital goods or more effective ones available for workers, and firms' investments in production methods and employees' skills; the last two can be included in the category of ‘human capital’. We model the role of these elements in the growth of a firm’s production and show that the ‘unobservable human capital’ is not a necessity for economic growth. Every factor that increases the productivity of inputs or reduces the costs of a firm contributes its growth.

3. Kinematics of Production

We denote by $q(t)\,(unit/y)$ the flow of production of a firm at time moment *t*. Unit y may be one day, two weeks etc. The accumulated production of the firm till time moment *t*, $Q(t)\ (unit)$, is:

$$Q(t) = Q(t_0) + \int_0^t q(s)ds, \quad Q'(t) = q(t), \quad Q''(t) = q'(t),$$

where $Q(t)\ (unit)$ is the accumulated production of the firm till time moment t, $Q'(t) = q(t)\,(unit/y)$ the momentous flow, and $Q''(t) = q'(t)\ (unit/y^2)$ the momentous acceleration of production at time t. This kinematics is a necessary prelude for production dynamics analogous with Newtonian mechanics[1].

4. Dynamization of the Neoclassical Theory

A common way to dynamize the static neoclassical theory of a firm is to assume that the firm maximizes its expected future profit. We omit uncertainty and assume the firm's profit function for time unit y as $\Pi(q(t),t) = P(t)q(t) - C(q(t))$, where $P(t)\ (\$/unit)$ is the price of the product of the firm, $q(t)\,(unit/y)$ the flow of production, and $C(q(t))\ (\$/y)$ the cost function. The dynamic optimization problem for the current value of the firm's profit from time $(0, t_1)$ is:

$$\max_{q(t)} \int_0^{t_1} F(q(t),t)\mathrm{d}t = \max_{q(t)} \int_0^{t_1} \mathrm{e}^{-rt}\Pi(q(t),t)\mathrm{d}t, \tag{1}$$

where e^{-rt} is discount factor with constant interest rate $r(1/y)$ (see Appendix Part *A*). The necessary condition for (1) – with possible boundary conditions – is the following Euler -equation:

$$\frac{\partial F}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial q'(t)}\right) = 0 \Rightarrow \mathrm{e}^{-rt}\frac{\partial \Pi}{\partial q} = 0 \Rightarrow \frac{\partial \Pi}{\partial q} = 0.$$

The necessary condition for (1) thus equals with that of static neoclassical theory. Dynamic optimization was used, however, to get an equation of motion for $q(t)$. The above shows that for this function $\Pi(q,t)$ should depend on $q'(t)$, but $q'(t)$ does not exist in profit functions in the

[1] A system of measurement units for economics is given in De Jong (1967). Measurement units are in parenthesis after the quantities.

literature of static neoclassical theory. Static neoclassical theory and its dynamization by dynamic optimization thus assume inconsistent profit functions (see Chiang (1992 pp. 49, 69, 292) and Avinash K. Dixit & Robert S. Pindyck (1994 pp. 254, 359)).

5. The Production Process of a Firm

We study the production of a one-product firm using labor, capital goods, and human capital (knowledge) as inputs. The production function is $q = f(L, K, R, t)$, where $q(unit/y)$ is the flow of production of the firm, $L(h/y)$ the labor input of the firm in ordinary production, $R(h/y)$ the labor input of the firm in the production of *human capital*, $K(\$)$ the value of the physical capital of the firm, and time t represents possible exogenous technological development. All partial derivatives of the production function are assumed positive.

If the physical capital of the firm is financed by loans, the interest costs are $rK(\$/y)$ where interest rate $r(1/y)$ is determined in the financial market. If capital is financed by equities, the interest costs rK are paid to the shareholders (equal r is assumed in both cases). Capital goods are assumed not to deteriorate, and the possible redemption of the loan capital is omitted; we can thus think that capital is financed by equities. For simplicity, perfect competition is assumed in the two labor markets so that the firm has no power concerning wages $w_1, w_2(\$/h)$ for these two types of labor. The profit of the firm from time unit y is:

$$\Pi(t) = P(t)f(L(t), K(t), R(t), t) - (1+\tau)[w_1(t)L(t) + w_2(t)R(t)] - r(t)K(t),$$

where τ is fixed assumed social security rate. The time derivative of the profit is:

$$\Pi'(t) = \frac{\partial \Pi}{\partial P}P'(t) + \frac{\partial \Pi}{\partial L}L'(t) + \frac{\partial \Pi}{\partial K}K'(t) + \frac{\partial \Pi}{\partial R}R'(t) + \\ + \frac{\partial \Pi}{\partial w_1}w_1'(t) + \frac{\partial \Pi}{\partial w_2}w_2'(t) + \frac{\partial \Pi}{\partial r}r'(t) + \frac{\partial \Pi}{\partial t}. \qquad (2)$$

Equation (2) shows that changes in the two wages and interest rate affect the profit, but these quantities are beyond control of the firm’s managers. Quantity $\partial\Pi/\partial t$ shows that elements like workers' learning, better education in the society etc. – that take place with time and create no costs for the firm – positively affect the firm's profitability. The variables the firm's managers can affect are the unit price, the two labor inputs, and the capital input. The adjustment rules for these quantities are:

$$x'(t) > 0 \quad if\ \frac{\partial \Pi}{\partial x} > 0, \quad x'(t) < 0 \quad if\ \frac{\partial \Pi}{\partial x} < 0,$$

and:

$$x'(t) = 0 \quad if\ \frac{\partial \Pi}{\partial x} = 0, \ \ x = P, L, K, R.$$

These rules make the first four additive terms on the right hand side of Eq. (2) non-negative, i.e. by adjusting in this way the managers increase the firm's profit with time. These adjustment rules are in line with the neoclassical theory that corresponds to situation $\partial \Pi / \partial x = 0,\ x = P, L, K, R$ with proper necessary conditions guaranteeing the state as the profit-maximizing one. We thus extend the neoclassical analysis by a dynamic adjustment that explains how the firm will reach its optimum in a stable case. Permanent growth and the possible collapse of the firm take place in non-stable cases.

A relation fulfilling the above rules is:

$$x'(t) = \Psi_x\left(\frac{\partial \Pi}{\partial x}\right),\ \Psi_x'\left(\frac{\partial \Pi}{\partial x}\right) > 0, \quad \Psi_x(0) = 0, \quad x = P, L, K, R. \tag{3}$$

The first order Taylor series approximation of function ψ_x in the neighborhood of the optimum $\partial \Pi / \partial x = 0$ is:

$$\Psi_x(y) \approx \Psi_x(0) + \Psi_x'\ (0)(y - 0) = \Psi_x'\ (0) y.$$

With this approximation, we can write Eq. (3) as:

$$x'(t) = \Psi_x'\ (0) \frac{\partial \Pi}{\partial x} \iff m_x x'(t) = \frac{\partial \Pi}{\partial x} \tag{4}$$

were:

$$\Psi_x'\ (0) = \frac{1}{m_x}.$$

Now, $L'(t)$ and $R'(t)$ with unit h / y^2 are the firm's acceleration of use of these two types of labor. Following Newton, we interpret the positive constants $m_x = 1 / \Psi_x'\ (0),\ \ x = L, R$ as the inertial factors (*masses*) of the adjusting quantities. These factors originate from the legislation for over-time work, the time needed to find skillful workers etc. The measurement unit $(\$ \times y^2) / h^2$ of factors $m_x,\ x = L, R$ makes Eq. (4) dimensionally well-defined, i.e. both sides have equal unit, see De Jong (1967).

With these assumptions, Eq. (4) with $x = L, R$ exactly corresponds to Newton's equation: $ma = F$ where $a = x'(t)$ and $F = \partial\Pi / \partial x$ with unit $\$ / h$.

Equation (4) with $x = P, K$ is not exactly the same as Newton's formulation because $P'(t)$ and $K'(t)$ are the flows of *P,K*, respectively. However, we still identify $\partial\Pi / \partial x$, $x = P, K$ with units *unit/y* and 1/*y*, respectively, as the *forces the firm's managers direct upon these quantities*, and constants $\psi'_x(0) = 1/m_x$, $x = P, K$ as the inverses of the inertial factors related to *P* and *K*. These inertial factors originate from the time and costs needed for price changes, installing new machines etc. The higher these factors, the greater are m_x, $x = P, K$ with units $unit^2$/\$ and1/\$, respectively. Weiss (1993), for example, find price inertia in concentrated industries.

System (4) is then:

$$\begin{pmatrix} m_P & 0 & 0 & 0 \\ 0 & m_L & 0 & 0 \\ 0 & 0 & m_R & 0 \\ 0 & 0 & 0 & m_K \end{pmatrix} \begin{pmatrix} P'(t) \\ L'(t) \\ R'(t) \\ K'(t) \end{pmatrix} = \begin{pmatrix} \partial\Pi / \partial P \\ \partial\Pi / \partial L \\ \partial\Pi / \partial R \\ \partial\Pi / \partial K \end{pmatrix}. \tag{5}$$

Estola (2001) shows that increasing demand and decreasing costs with time increase the production of a profit-seeking firm. We can thus omit these cases and so we assume the unit price to stay fixed, $P'(t) = 0$. The block diagram of model (5) is in figure 1. It shows how the managers of the firm control the inputs L, R, K in a closed-loop way, and labor, financial, and product markets open the system for external effects via quantities w_1, w_2, r, P. Public sector may act as an open-loop controller of the system by parameter τ (see Marshall (1978)).

In the following we omit uncertainties to keep the modeling simple. However, uncertainty can be added in the analysis by replacing the quantities in the firm's managers' decision-making by their expected values (see Estola (2001)).

6. Increasing Returns to Scale in Labor and Capital

First we show that growth may occur due to increasing returns to scale in the traditional inputs. We assume a Cobb-Douglas type of production function:

$$q(t) = f(L(t), K(t)) = A\left(\frac{L(t)}{L_0}\right)^{\alpha}\left(\frac{K(t)}{K_0}\right)^{\beta},$$

where A with unit *unit/y* is a technology constant, L_0, K_0 with units h/y and \$, respectively, the values of the two inputs at time 0, and α, β positive numbers. In this form the function is dimensionally homogeneous, and the profit function with unit $\$/y$ is:

$$\Pi(t) = PA\left(\frac{K(t)}{K_0}\right)^{\beta}\left(\frac{L(t)}{L_0}\right)^{\alpha} - (1+\tau)w_1 L(t) - rK(t).$$

The adjustment system for the two inputs is:

$$\begin{pmatrix} m_L & 0 \\ 0 & m_K \end{pmatrix}\begin{pmatrix} L'(t) \\ K'(t) \end{pmatrix} = \begin{pmatrix} \partial\Pi/\partial L \\ \partial\Pi/\partial K \end{pmatrix}, \tag{6}$$

where the components of the force vector $F = (\partial\Pi/\partial L, \partial\Pi/\partial K)$ are:

$$\frac{\partial\Pi}{\partial L} = \frac{\alpha AP}{L_0}\left(\frac{L(t)}{L_0}\right)^{\alpha-1}\left(\frac{K(t)}{K_0}\right)^{\beta} - (1+\tau)w_1,$$

$$\frac{\partial\Pi}{\partial K} = \frac{\beta AP}{K_0}\left(\frac{L(t)}{L_0}\right)^{\alpha}\left(\frac{K(t)}{K_0}\right)^{\beta-1} - r.$$

The force vector shows how the exogenous variables P, w_1, r affect the adjustment, and that government can affect the system by parameter τ. Due to the non-linearity of System (6), its dynamic behavior is studied by phase diagrams. The slopes of the demarcation lines $L'(t) = K'(t) = 0$ in coordinate system (L, K) are:

$$\left.\frac{\mathrm{d}K}{\mathrm{d}L}\right|_{L'(t)=0} = \frac{(1-\alpha)}{\beta}\frac{K}{L}, \quad \left.\frac{\mathrm{d}K}{\mathrm{d}L}\right|_{K'(t)=0} = \frac{\alpha}{(1-\beta)}\frac{K}{L}, \quad L, K > 0.$$

The attract/repel – character of the demarcation lines depends on the following partials:

$$\frac{\partial L'(t)}{\partial L} = \frac{\alpha(\alpha-1)AP}{L_0^2 m_L}\left(\frac{K(t)}{K_0}\right)^{\beta}\left(\frac{L(t)}{L_0}\right)^{\alpha-2},$$

$$\frac{\partial K'(t)}{\partial K} = \frac{\beta(\beta-1)AP}{K_0^2 m_K}\left(\frac{L(t)}{L_0}\right)^{\alpha}\left(\frac{K(t)}{K_0}\right)^{\beta-2}.$$

The behavior of the system thus critically depends on the values of α, β (see Appendix Part B).

Case 1: 0<α, β<1. In this case both inputs obey decreasing returns to scale. Both demarcation lines are then upward sloping in coordinates (*L*,*K*), and the corresponding phase diagrams are in figures 2 and 3. The equilibrium state $L'(t)=K'(t)=0$ is a sink in figure 2 $(\alpha+\beta<1)$ and a saddle in figure 3 $(\alpha+\beta>1)$. Figure 2 displays a dynamic extension for the neoclassical theory. Figure 3, on the other hand, shows that permanent growth in the use of the inputs (and also in q through the production function) is possible when the two inputs together obey increasing returns. If the initial values L_0, K_0 are great enough, permanent growth takes place while small L_0, K_0 lead to the collapse of the firm. The profit of a growing firm will increase with time.

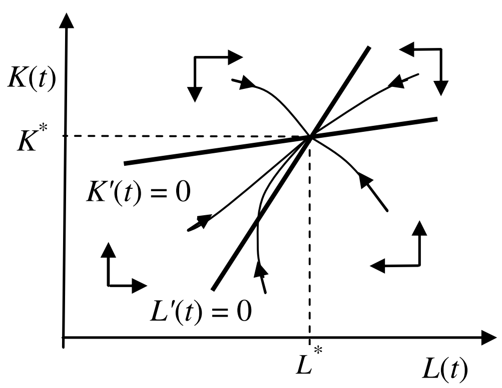

Figure 2. Decreasing returns in both inputs.

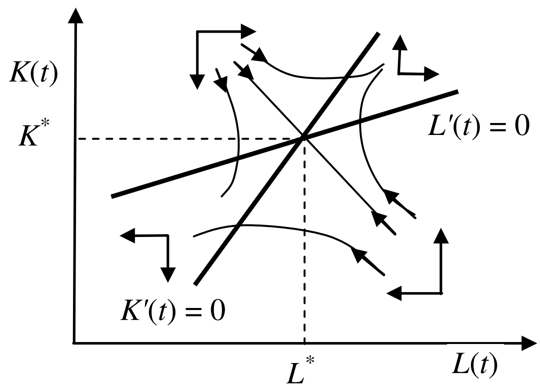

Figure 3. Increasing returns in the two inputs together.

Case 2: α<1, β>1. Here labor obeys decreasing and capital increasing returns to scale. Line $L'(t)=0$ is now upward and line $K'(t)=0$ downward sloping in coordinates (L, K). The equilibrium is a saddle (Fig. 4). Permanent growth is possible if the initial values L_0, K_0 are great enough while low values for L_0, K_0 lead to the collapse of the firm. Only firms with K_0 great enough can survive and their profitability increases with time.

Case 3: α,β>1. Here capital and labor both obey increasing returns to scale. Both demarcation lines are now downward sloping in coordinates (L, K). The only possible phase diagram is in figure 5 because the case line $L'(t)=0$ steeper implies $\alpha+\beta<1$, which is impossible. The equilibrium in figure 5 is a saddle. Permanent growth is possible if L_0, K_0 are great enough while low values for L_0, K_0 lead to the collapse of the firm. If L_0, K_0 are great enough, the firm grows permanently and increases its profitability with time.

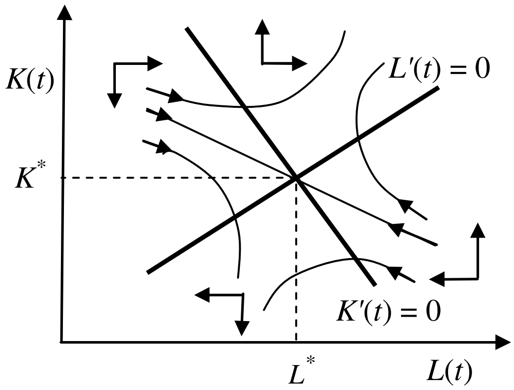

Figure 4. Increasing returns in capital and both inputs.

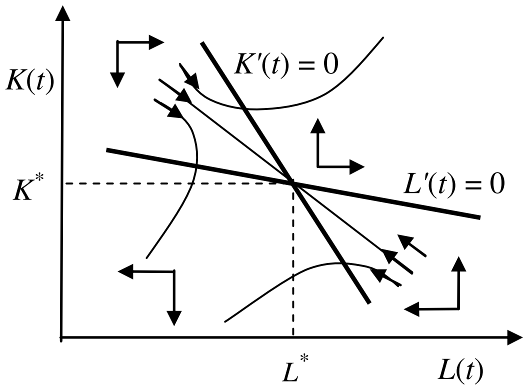

Figure 5. Increasing returns in both inputs.

We omitted the case $\alpha > 1, \beta < 1$ because it is symmetric to that in Case 2. In figures 2-5, changes in exogenous variables P, w_1, r, τ move the positions of the demarcation lines. If the initial state of the system is the equilibrium, a move in one demarcation line starts an adjustment toward the new equilibrium state in figure 2 and away from the new equilibrium in figures 3-5, unless the initial point happens to be on a saddle-path.

7. Increasing Returns to Scale in Human Capital

Next we omit physical capital and concentrate on the role of human capital in the production process. The production function of the firm is assumed as:

$$q(t) = BH(t)\left(\frac{L(t)}{L_0}\right)^{\alpha}, \qquad H(t) = D\left(\frac{R(t)}{R_0}\right)^{\gamma}$$

where $L(h/y)$ is the labor input as earlier, $R(h/y)$ the labor input in the production of human capital H(\$/y), and $B(unit/\$), D(\$/y), a, y$ are positive constants. Notice that H is the flow – and not the level – of human capital, and only increases in human capital are assumed to affect q.

Human capital is measured by its monetary value when sold to other firms. The profit function is then:

$$\Pi(t) = PA\left(\frac{R(t)}{R_0}\right)^{\gamma}\left(\frac{L(t)}{L_0}\right)^{\alpha} - (1+\tau)[w_1 L(t) + w_2 R(t)], \quad A = BD.$$

This model is identical as in the previous section when R is replaced by K, γ by β, and $(1+\tau)w_2$ by r. Again, either of the two types of labor may obey increasing returns to scale, or they may together cause increasing returns to scale in the production. We could have added in this analysis physical capital, too, and get results where every combination of the three inputs may cause increasing returns to scale in the production. Other forms for the production function could also have been assumed with similar results.

The theories of Lucas (1988) and Romer (1990) base on increasing returns to scale in production due to human capital. However, our modeling shows that the somewhat unclear concept of human capital is not a necessity for economic growth. Similar results are obtained with increasing returns for any input, or because several inputs together obey increasing returns. The block diagram in figure 1 gives a hint of how a more detailed description of the production process could reveal elements where efficiency can be increased.

8. Conclusions

In the static neoclassical theory of a firm, time is abstracted away, and increasing returns to scale are not allowed. A different framework is thus needed for modeling economic growth at micro level. As a candidate we introduced a dynamic extension for the static neoclassical theory, analogous with the Newtonian formulation in physics. The constructed models are of control theoretic nature that is demonstrated by a block diagram in figure 1. Our main results are: 1) Permanent growth may occur due to increasing returns to scale in any input or a combination of inputs of a firm. 2) The input mix of a growing firm will change with time toward greater profitability. 3) The managers of a profit-seeking firm control the production process in a closed-loop way, and government may act with taxes as an open-loop controller of the process. A decrease in interest rate, if controllable by the monetary authority, can also be used in promoting growth as the defined force vector shows.

In real life there exist factors that slow down and eventually stop the growth of a firm, like increasing wage level, the saturation of the market, the appearance of competing firms, etc. However, we have evidence of long-lasting growth of corporations like IBM, HP, Nokia etc. Even though these growth processes may not last forever, we believe that it is more meaningful to model these as explosive processes rather than to assume that these firms have been approaching their equilibrium states last 20 years, as the neoclassical theory claims. We believe that there are elements in the production processes of these firms that allow them to grow in a profitable way, which elements we modeled here. However, because firms' production methods change with time, one specific model may not keep its accuracy too long.

Appendix

Part A: Continuous time interest rate $r(1/y)$ is defined as the growth rate of invested money $x(t)(\$)$:

$$r = \frac{x'(t)}{x(t)} \Longleftrightarrow x'(t) = rx(t) \Rightarrow x(t) = x(t_0)\mathrm{e}^{r\times(t-t_0)}.$$

The unit $\$/y$ of $x'(t)$, where y is a time unit, explains that of r. The last equation is the solution of the differential equation with fixed r. With $t_0 = 0$, the interest factor is e^{rt} where time t has unit y. The exponent of e is thus dimensionless.

Part B: The necessary conditions for the critical point $\partial\Pi/\partial L = = \partial\Pi/\partial K = 0$ to maximize the profit are:

$$\frac{\partial\Pi}{\partial L} = \frac{\alpha AP}{L_0}\left(\frac{L}{L_0}\right)^{\alpha-1}\left(\frac{K}{K_0}\right)^{\beta} - (1+\tau)w_1 = 0,$$

$$\frac{\partial\Pi}{\partial K} = \frac{\beta AP}{K}\left(\frac{L}{L_0}\right)^{\alpha}\left(\frac{K}{K_0}\right)^{\beta-1} - r = 0.$$

The sufficient condition for profit maximization is that the first minor determinant of the Hessian matrix is negative, and the second is positive. The Hessian matrix is:

$$H = \begin{pmatrix} \partial^2\Pi/\partial L^2 & \partial^2\Pi/\partial L\partial K \\ \partial^2\Pi/\partial K\partial L & \partial^2\Pi/\partial K^2 \end{pmatrix}.$$

The first minor of H is $H_1 = \partial^2\Pi / \partial L^2$, and the second minor equals the determinant of the Hessian $|H|$ is:

$$|H| = \frac{\partial^2\Pi}{\partial L^2}\frac{\partial^2\Pi}{\partial K^2} - \left(\frac{\partial^2\Pi}{\partial L \partial K}\right)^2,$$

were:

$$\frac{\partial^2\Pi}{\partial L^2} = \frac{AP\alpha(\alpha-1)}{L_0^2}\left(\frac{K}{K_0}\right)^{\beta}\left(\frac{L}{L_0}\right)^{\alpha-2},$$

$$\frac{\partial^2\Pi}{\partial K^2} = \frac{AP\beta(\beta-1)}{K_0^2}\left(\frac{K}{K_0}\right)^{\beta-2}\left(\frac{L}{L_0}\right)^{\alpha},$$

and:

$$\frac{\partial^2\Pi}{\partial K \partial L} = \frac{AP\alpha\beta}{L_0 K_0}\left(\frac{K}{K_0}\right)^{\beta-1}\left(\frac{L}{L_0}\right)^{\alpha-1}.$$

Thus: $|H| = \dfrac{\alpha\beta P^2 A^2}{L_0^2 K_0^2}\left(\dfrac{L}{L_0}\right)^{2(\alpha-1)}\left(\dfrac{K}{K_0}\right)^{2(\beta-1)}[(\alpha-1)(\beta-1)-\alpha\beta].$

If $0 < \alpha < 1$ then $H < 0$. On the other hand, requirement $|H| > 0$ corresponds to $\alpha + \beta < 1$. Thus for the critical point to be a maximum – as is assumed in the neoclassical framework – neither of the two inputs separately or together can obey increasing returns to scale. The stability of the system, too, depends on the Hessian matrix, see for example McCafferty (1990 p. 138). The trace of the Hessian matrix is:

$$Tr(H) = \frac{\partial^2\Pi}{\partial L^2} + \frac{\partial^2\Pi}{\partial K^2} = AP\left(\frac{L(t)}{L_0}\right)^{\alpha}\left(\frac{K(t)}{K_0}\right)^{\beta}\left(\frac{\alpha(\alpha-1)}{L^2} + \frac{\beta(\beta-1)}{K^2}\right).$$

The system is stable if $|H| > 0 \iff (\alpha-1)(\beta-1) > \alpha\beta$ i.e. $\alpha+\beta<1$ and $Tr(H) < 0$, which occurs certainly if $0 < \alpha, \beta < 1$; see (Fig. 2).

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