THE CHAOTIC REAL INCOME GROWTH MODEL

Vesna JABLANOVIC*

Abstract. *This paper examined an economy in which investment is an increasing function of previous period's income, and an increasing function of current capital stock. Also, it is supposed that profit is a decreasing function of the capital stock. Also, it is supposed that consumption is a quadratic function of the past real income.*

The basic aim of this paper is to construct a relatively simple chaotic real income growth model that is capable of generating stable equilibria, cycles, or chaos. The constant depreciation rate and the coefficients of the profit, capital stock and investment functions determine the local stability of the real income growth.

Keywords: *Capital Investment Profit Real income Chaos.*

1. Introduction

The basic aim of this analysis is to provide a relatively simple chaotic economic growth model.

Chaos theory reveals structure in aperiodic, dynamic systems. The number of nonlinear business cycle models use chaos theory to explain complex motion of the economy. Chaotic systems exhibit a sensitive dependence on initial conditions: seemingly insignificant changes in the initial conditions produce large differences in outcomes. On the other hand, in the stable dynamic systems, a small change in one variable produces a small and easily quantifiable systematic change.

Chaos theory started with Lorenz's [1963] discovery of complex dynamics arising from three nonlinear differential equations leading to turbulence in the weather system. Li and Yorke [1975] discovered that the simple logistic curve can exibit very complex behaviour. Further, May [1976] described chaos in population biology. Chaos theory has been applied in economics by Benhabib and Day [1981, 1982], Day [1982, 1983], Baumol W. & Benhabib, J. [1989], Grandmont [1985], Goodwin [1990], Medio [1993], Lorenz [1993], Jablanovic [2011 2013, 2016], Puu, T. [2003], Zhang W. B. [2006], among many others.

* University of Belgrade, Faculty of Agriculture, Serbia, vesnajab@ptt.rs

2. The model

The chaotic economic growth model is presented by the following equations:

$$\pi_t = -\beta K_t \quad \beta > 0 \tag{1}$$

$$\Delta K = \alpha \pi_t + A \qquad \alpha > 0 \tag{2}$$

$$\Delta K = I_t - \delta K_t \quad \delta > 0 \tag{3}$$

$$I_t = \eta Y_{t-1} - \mu \; K_t \qquad \eta > 0, \quad \mu > 0 \tag{4}$$

$$Y_t = C_t + I_t \tag{5}$$

$$C_t = \gamma \; Y_{t-1} \quad \gamma > 0 \tag{6}$$

with Y – real income, K – capital stock, I – investment, C – consumption, A – an autonomous investment, π – profit, δ – constant depreciation rate, structural constants: α – the coefficient of capital stock function, β – the coefficient of profit function, η and μ – the coefficients of investment function, γ – the coefficient of consumption function.

(1) profit is a decreasing function of the capital stock; (2) shows that an autonomous investment and profit are factors of the capital stock change; (3) shows how the capital stock changes over time. Here δ is the rate of physical depreciation so that between year t and year $t+1$, δK_t units of capital are lost from depreciation. But during year t, there is investment (I_t) that yields new capital in the following year; (4) investment is an increasing function of previous period's income, and a decreasing function of current capital stock; (5) shows national accounting identity; (6) consumption is a quadratic function of the past real income.

It is supposed that $A = 0$. Now, putting (1), (2), (3), (4), (5) and (6) together we immediately get:

$$Y_t = \left[\frac{(\delta - \alpha\,\beta)\,\eta}{\mu + \delta - \alpha\,\beta}\right] Y_{t-1} + \left[\frac{(\delta - \alpha\,\beta)\,\gamma + \mu\,\gamma}{\mu + \delta - \alpha\,\beta}\right] Y_{t-1}{}^2. \tag{7}$$

Further, it is assumed that the current value of the real income is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the real income growth rate depends on the current size of the real income, Y, relative to its maximal size in its time series Y^m. We introduce y as $y = Y / Y^m$. Thus y range between 0 and 1. Again we index y by t, i.e., write y_t to refer to the size at time steps $t = 0,1,2,3,...$ Now growth rate of the real income is measured as

$$y_t = \left[\frac{(\delta - \alpha\,\beta)\,\eta}{\mu + \delta - \alpha\,\beta}\right] y_{t-1} + \left[\frac{(\delta - \alpha\,\beta)\,\gamma + \mu\,\gamma}{\mu + \delta - \alpha\,\beta}\right] y_{t-1}{}^2. \tag{8}$$

This model given by equation (8) is called the logistic model. For most choices of α, β, γ, δ, η, and μ there is no explicit solution for (8). Namely, knowing α, β, γ, δ, η, and μ measuring y_0 would not suffice to predict y_t for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect – the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

3. The logistic equation

It is possible to show that iteration process for the logistic equation

$$z_t = \pi z_{t-1}(1 - z_{t-1}),\ \pi \in [0,4],\ z_{t-1} \in [0,1] \tag{9}$$

is equivalent to the iteration of growth model (8) when we use the identification

$$z_{t-1} = -\left[\frac{(\delta - \alpha\,\beta)\,\gamma + \mu\,\gamma}{(\delta - \alpha\,\beta)\,\eta}\right] y_{t-1} \text{ and } \pi = \left[\frac{(\delta - \alpha\,\beta)\,\eta}{\mu + \delta - \alpha\,\beta}\right] \tag{10}$$

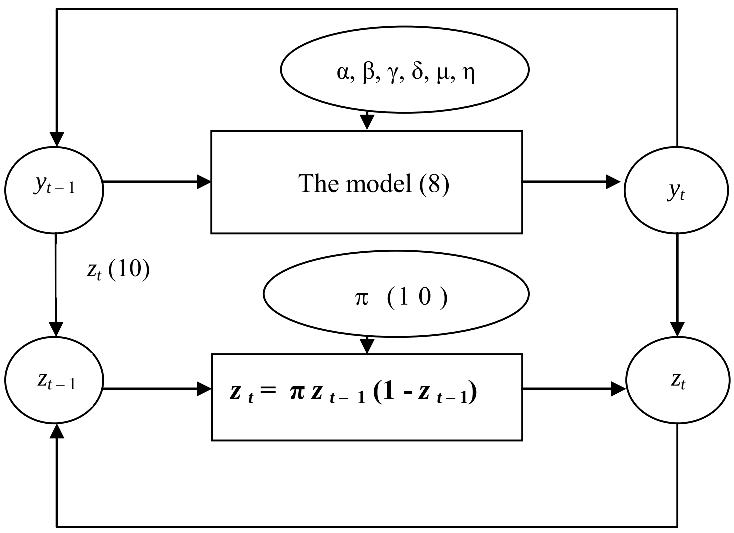

Figure 1. Two quadratic iteratiors running in phase are tightly coupled by the transformations indicated.

Using (8) and (9) we obtain:

$$z_t = -\left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{(\delta-\alpha\,\beta)\,\eta}\right] y_t =$$

$$= -\left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{(\delta-\alpha\,\beta)\,\eta}\right]\left\{\left[\frac{(\delta-\alpha\,\beta)\,\eta}{\mu+\delta-\alpha\,\beta}\right] y_{t-1} + \left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{\mu+\delta-\alpha\,\beta}\right] {y_{t-1}}^2\right\} =$$

$$= -\left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{\mu+\delta-\alpha\,\beta}\right] y_{t-1} - \left\{\frac{[(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma]^2}{(\delta-\alpha\,\beta)\,(\mu+\delta-\alpha\,\beta)\,\eta}\right\} {y_{t-1}}^2.$$

On the other hand, using (9) and (10) we obtain:

$$z_t = \pi z_{t-1}(1 - z_{t-1}) =$$

$$= -\left[\frac{(\delta-\alpha\,\beta)\,\eta}{(\mu+\delta-\alpha\,\beta)}\right]\left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{(\delta-\alpha\,\beta)\,\eta}\right] y_{t-1}\left\{1 + \left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{(\delta-\alpha\,\beta)\,\eta}\right] y_{t-1}\right\} =$$

$$= -\left[\frac{(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma}{\mu+\delta-\alpha\,\beta}\right] y_{t-1} - \left\{\frac{[(\delta-\alpha\,\beta)\,\gamma+\mu\,\gamma]^2}{(\delta-\alpha\,\beta)\,(\mu+\delta-\alpha\,\beta)\,\eta}\right\} {y_{t-1}}^2.$$

Thus we have that iterating (8) is really the same as iterating (9) using (10). It is important because the dynamic properties of the logistic equation (9) have been widely analyzed (Li and Yorke (1975), May (1976)).

It is obtained that:

(i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$;

(ii) For $1 < \pi < 3{,}57$ there exist fixed points the number of which depends on π;

(iii) For $1 < \pi < 2$ all solutions monotnically increase to $z = (\pi - 1) / \pi$;

(iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1) / \pi$;

(v) For $3 < \pi < 4$ all solutions will continously fluctuate;

(vi) For $3{,}57 < \pi < 4$ the solution become "chaotic" wihch means that there exist totally aperiodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever.

Important parameter π values “0, 1, 1, 2, 3” are part of the Fibonacci sequence. The Fibonacci Sequence is the series of numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... There is an interesting pattern: The Fibonacci Sequence is found by adding the two numbers before it together. That is found by adding the two numbers before it (0 + 1). The 2 is found by adding the two numbers before it (1 + 1). The 3 is found by adding the two numbers before it (1+2). Namely, each number is the sum of the two numbers before it. If we make squares with those widths, we get a nice spiral. Also, if we take any two successive, important values of parameter π ("2, 3"), their ratio is very close to the Golden ratio which is approximately 1.618034... For example 3/2 is 1.5. The golden ratio has the approximate value of 1.618. The golden ratio and the golden rectangle are connected. This is because the ratio of the longer side of a golden rectangle to the shorter side is equal to the golden ratio ($1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + \ldots$) (Jablanovic, 2016, pp. 38-39).

4. Conclusion

This paper creates the chaotic real income growth model (8). This difference equation (8) will possess a chaotic region. Two properties of the chaotic solution are important: firstly, given a starting point y_0 the solution is highly sensitive to variations of the parameters α, β, γ, δ, η, and secondly, given the parameters α, β, γ, δ, η, the solution is highly sensitive to variations of the initial point y_0. In both cases the two solutions are for the first few periods rather close to each other, but later on they behave in a chaotic manner.

A key hypothesis of this work is based on the idea that the coefficient $\pi = \left[\frac{(\delta - \alpha\,\beta)\,\eta}{\mu + \delta - \alpha\,\beta}\right]$ plays a crucial role in explaining local stability of the real income, where, δ is the constant depreciation rate, α – the coefficient of capital stock function, β – the coefficient of profit function, η and μ – the coefficients of investment function.

REFERENCES

[1] Baumol W. & Benhabib, J. (1989), Chaos: Significance, Mechanism, and Economic Application, *The Journal of Economic Perspectives*, Vol. **3,** No. **1,** pp. 77-105.

[2] Benhabib, J., & Day, R. H. (1981), Rational Choice and Erratic Behaviour, *Review of Economic Studies* 48: pp. 459-471.

[3] Benhabib, J., & Day, R. H. (1982), Characterization of Erratic Dynamics in the Overlapping Generation Model, *Journal of Economic Dynamics and Control* 4: pp. 37-55.

[4] Day, R. H. (1982), Irregular Growth Cycles, *American Economic Review* 72: pp. 406-414.

[5] Day, R. H. (1983), The Emergence of Chaos from Classica Economic Growth, *Quarterly Journal of Economics* 98, pp. 200-213.
[6] Goodwin, R. M. (1990), *Chaotic Economic Dynamics*, Clarendon Press, Oxford.
[7] Grandmont, J. M. (1985), On Enodgenous Competitive Business Cycles, *Econometrica* 53, pp. 994-1045.
[8] Jablanovic, V. (2011), *Budget Deficit and Chaotic Economic Growth Models*. Aracne editrice S.R.L, Roma.
[9] Jablanovic, V. (2013), *Elements of Chaotic Microeconomics*, Aracne editrice S.R.L., Roma.
[10] Jablanovic, V. (2016), *A Contribution to the Chaotic Economic Growth Theory*. Aracne editrice S.R.L., Roma.
[11] Li, T., &Yorke, J. (1975), Period Three Implies Chaos, *American Mathematical Monthly* 8; pp. 985-992.
[12] Lorenz, E. N. (1963), Deterministic nonperiodic flow, *Journal of Atmospheric Sciences* 20; 130-141.
[13] Lorenz, H. W. (1993), *Nonlinear Dynamical Economics and Chaotic Motion*, 2nd edition, Springer-Verlag, Heidelberg.
[14] May, R. M. (1976), Mathematical Models with Very Complicated Dynamics, *Nature* 261, pp. 459-467.
[15] Medio, A. (1993), *Chaotic Dynamics: Theory and Applications to Economics,* Cambridge University Press, Cambridge.
[16] Puu, T. (2003), *Attractors, Bifurcations, and Chaos – Nonlinear Phenomena in Economics*, Springer.
[17] Zhang W. B. (2006), *Discrete Dynamical Systems, Bifurcations and Chaos in Economics*, Elsevier B. V.