ECONOPHYSICS Section

NEWTONIAN AND LAGRANGIAN MECHANICS OF A PRODUCTION SYSTEM

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Abstract. *We define the economic correspondents for kinetic and potential energy of production and mutually coherent Newtonian and Lagrangian frameworks for modeling production. The neoclassical theory is shown to correspond to zero-force situation in the introduced framework where the potential energy of the production system is in its minimum. This explains why it has been difficult to explain real world production dynamics by using the neoclassical framework. Our framework adds dynamics in the neoclassical theory and includes in it cases like firms' permanent growth, business cycles, and bankruptcies. These are impossible events in the neoclassical framework that assumes firms to produce at constant positive profit maximizing flow of production.* ***JEL: D21, C62, O12.***

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1. Introduction

Neoclassical economics was born when William Stanley Jevons, Alfred Marshall, and Leon Walras, among others, started to use the formalism of classical mechanics in modeling the purposeful behavior of human beings, [1]. The latter was suggested by Adam Smith as the basis for the science. The neoclassical framework combines these two ways of thinking and it is based on the concept of static equilibrium. In spite of numerous attempts, the pioneers of the neoclassical theory were not able to define the forces acting in economies and the economic energy concepts so that the formalism of classical mechanics could have been applied also in modeling dynamic economic events.

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MasColell et al. [2] p. 620 state the problems in the neoclassical framework as follows: "*A characteristic feature that distinguishes economics from other scientific fields is that, for us, the equations of equilibrium constitute the center of our discipline. Other sciences, such as physics or even ecology, put comparatively more emphasis on determination of dynamic laws of change. … The reason, informally speaking, is that economists are good (or so we hope) at recognizing a state of equilibrium but are poor at predicting precisely how an economy in disequilibrium will evolve. Certainly there are intuitive dynamic principles: if demand is larger than supply then price will increase, if price is larger than marginal cost then production will expand… The difficulty is in transforming these informal principles into precise dynamic laws*".

Although Mirowski [1] is critical for the analogy between economics and physics, he admits that the use of the methods of classical mechanics was essential for economics to become a respected science. The concept of energy was popular among the pioneers of neoclassical economics because it unified the modeling principles in physics into a single principle. In these works of economics by e.g. Nicholas-Francois Canard, William Stanley Jevons, Hermann Gossen, Irwing Fisher, Leon Walras, and Paul Samuelson, the concept corresponding to potential energy was utility, and that corresponding to kinetic energy was total expenditures of consumers (*ibid*, pp. 223-227).

Our opinion of these matters coincides with the following views. Joseph Schumpeter[1]: ..."*We must not copy out actual arguments but we can learn from physics how to build up an exact argument. ... there are obviously a set of concepts and procedures which … are of so general a character as to be applicable to an indefinite number of different fields. The concepts of Potential or Friction of Inertia are of that kind…*" Solow [3] pp. 330-1: "*My impression is that the best and the brightest of the profession proceed as if economics is the physics of society. If the project of turning economics into a hard science could succeed, then it would surely be worth doing*". Walras [4] p. 71: "*...the pure theory of economics is a science which resembles the physico-mathematical sciences in every respect*".

[1] In a letter to Edwin Bidwell, Wilson, 19 May 1937, in Harward University Archives, Wilson Correspondence, HUG 4878.203.

Samuelson [5] p. 355 describes Fisher [6] as "*perhaps the best of all doctoral dissertations in economics*". Fisher [6] is the first published work where the correspondences between physics and economics are explicitly defined. Fisher translated the main physical concepts into economics as follows: *particle* = *individual*, *space* = *commodity*, *force* = *marginal utility*, *work* = *disutility*, *energy* = *utility*. These correspondents did not turn out to be useful for economics, however, which explains the critical attitude of [1] on this analogy. In section 4 we alter these definitions, however, and show that if defined correctly this analogy solves many current problems in economics.

Mirowski [1] criticizes economists about copying classical mechanics and not utilizing 20th century physics. However, economics cannot jump into utilizing quantum mechanics before it has been able to apply even the simplest principles of modeling dynamic phenomena in physics that still work in most macro level events. Because neoclassical economics is based on a framework invented in physics, we can project that besides classical mechanics also other principles of modeling in physics, like classical statistical mechanics and quantum mechanics, have fields of application in economics. On the other hand, Lagrangian and Hamiltonian formalisms are potential frameworks for modeling in economics too. These principles require, however, that a framework for modeling economic phenomena analogous to Newtonian mechanics has been defined.

The type of research described above is currently made under the field known as *econophysics*. Thus *econophysics is not dissent thinking in economics* but rather critical rethinking of the foundations of the neoclassical framework. A majority of research in econophysics applies the tools developed in classical statistical mechanics in modeling systems consisting of various interacting economic units, see e.g. [7]. Thus econophysics applies the modern methods in physics developed for modeling complex phenomena, as insisted in [1]. The problem in econophysics is, however, that a bridge between static equilibrium analysis in neoclassical economics and stochastic dynamic models in econophysics has not been presented. Our aim here is to fill this gap and to replace the informal dynamic principles in the neoclassical framework mentioned earlier in [2] by precise dynamic laws. One inspiration for this study was the identification of money as the economic correspondence for energy in physics, see e.g. [8-10]. We define the kinetic and the potential energy of a production system as monetary quantities in a consistent way with these authors, and in this way we can define the Lagrangian mechanics of a production system.

1.1. Motivation

The analogy in methods of modeling in physics and in economics can be justified as follows. Economic units have a free will to operate as they wish. However, this free will is restricted by the willingness of economic units to attain the goals they have set for themselves, which forces them to behave consistently with attaining these goals. This kind of behavior can be called *rational*. In Aristotelian physics, particles were thought to have an objective or *telos* to move along a certain path, [11]. Later Newton explained the *telos* of a particle by the forces acting upon it. Hamilton's principle, on the other hand, states that the motion path of a particle minimizes its energy. We can thus explain the motion of particles within the Aristotelian framework so that they have a "will" (*telos*) to reach the minimum point of their potential energy.

The above explains why the frameworks for modeling developed in physics can be applied in economics too. *We can model the behavior of economic units so that they are assumed to have a will to reach a certain state, like maximal utility or profit (minimum potential energy i.e. minimum deviation from the optimal state)*. Economic units, too, face several kinds of interactions like marketing. The Lagrangian and Hamiltonian frameworks for modeling are thus analogous with neoclassical economics if we assume that economic units are not always in their optimum, but due to different kinds of interactions the units tend to improve their current state to reach their optimum. This explains why it has been difficult to model observed changes and economic growth in the neoclassical framework where economic units are assumed to behave in their optimal way. In physics, this would correspond to the assumption that a particle has reached its minimum state of potential energy where it does not "want" to change its state of motion any more.

The problem in the neoclassical framework can be explained by the following example. Let us consider an empty bowl and a marble on its edge. If the marble is let loose, it will end in the bottom of the bowl via the trajectory that can be solved by using either Newtonian or Lagrangian framework. Once the trajectory is solved, we know the position of the marble and its direction of motion at every instant of time during the motion. If this dynamic phenomenon would be solved by using the neoclassical framework, however, we could only say that after some time the marble will be immobile in the bottom of the bowl. Thus by using the neoclassical framework we could not forecast whether there is overshooting in the adjustment or not, where the marble is and in which direction it is moving at every instant of time, and when the marble will reach its equilibrium. The neoclassical framework is useless e.g. in explaining and forecasting the transformation process of former centrally planned economies to market economies. What the neoclassical framework can forecast is to say that after roughly 40 years the former "eastern countries" have moved from the "centrally planned equilibrium" to the "market equilibrium". This framework does not give any tools to forecast or guide this process.

The neoclassical framework has though been dynamized by applying dynamic optimization, see e.g. [12] or [13]. It is shown in [14], however, that the static and dynamic neoclassical frameworks are inconsistent with each other because they assume different target functions for economic units. Thus both these frameworks cannot be accepted simultaneously. Our framework here, on the other hand, gives the static neoclassical one as a special case and thus these frameworks are consistent.

According to [11], Aristotelian teleology was applied to humans as well. Humans were assumed to have a *telos to behave in a rational way*. Following Newton, we can explain the regularities caused by the intentional behavior of economic units by defining the "forces" created by economic units that are acting upon economic quantities. In economics, however, this kind of framework has not been accepted. Thus economics has a long way to go before the science has utilized all the knowledge developed in physics for modeling complex phenomena by applying mutually coherent principles.

2. Economic concepts corresponding to the physical ones

2.1. Kinematics

We agree with [6] that economic kinematics can be described as the position of an economic quantity in a coordinate system and according to the movement of the point. The only change we make is to assume that there exist factors resisting changes in these quantities, and we model this inertia according to physics by defining the inertial *"masses"* of economic quantities. Thus we model in economics the dynamics of "*particles with a mass*" in different coordinate systems. The kinematics of production (consumption) of good *i* can be expressed as

$$Q_i(t) = Q_i(t_0) + \int_{t_0}^{t} q_i(s)ds\,, \qquad t > t_0,$$
$$\dot{Q}_i(t) = q_i(t), \qquad \ddot{Q}_i(t) = \dot{q}_i(t),$$

where $Q_i(t)$ $(unit), Q_i(t_0)$ $(unit)$ are the accumulated amounts of production (consumption) of good i at time moments t, t_0, respectively, $\dot{Q}_i(t) = q_i(t)$ $(unit/y)$ the momentous flow of accumulated production (consumption), and $\ddot{Q}_i(t) = \dot{q}_i(t)$ $(unit/y^2)$ the momentous acceleration of accumulated production (consumption) at time moment t. Unit y is time, and it can be e.g. a year or a week (compare with the units of speed m/s and acceleration m/s^2 in physics[2]), and by s is denoted running time during (t_0, t). Separate notation q is chosen for the flow of production because in economics this is a fundamental quantity while in physics, the corresponding fundamental quantity is position or the accumulated length of motion of a body. Accumulated production is needed in economics e.g. in measuring capital stocks and in modeling learning at work due to work experience, and accumulated consumption is needed e.g. in measuring the amount of calories a consumer has received by eating.

2.2. The concepts of economic energy

We define the economic correspondents for the two energy concepts in physics in a new way, and we show that these definitions are useful in modeling economic dynamics. It is essential that the economic correspondent for energy has the same characteristics as energy has in physics, that is, it is the initiator of motion of the system. Without external energy, a physical system that has ended up into its minimum level of energy cannot move to a state of higher energy. With these arguments, we choose *money for the dimension of the concepts of economic energy* and we measure it in unit[3] *euro*.

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[2] A system of measurement units for economics is given in [15].

[3] Measurable quantities have a certain *dimension* (e.g. "length" in physics), but one dimension may contain several measurement units (the units of length are e.g. *meter, kilometer, and inch*.)

In an economy, by money you can get almost everything and all prices are expressed in monetary units. In physics, "price" is usually expressed in units of energy. If a physical system moves from one state to another, this movement either uses or releases energy unless the two states have the same level of energy. Let us suppose a consumer with the economic state that contains a yacht. This consumer moves to a state where the yacht has been replaced by a car with other things being equal (except perhaps cash). This change may have used or released money, or the exchange between the yacht and the car has been made without monetary transaction in the case their values were equal.

2.2.1. Kinetic energy of production

The kinetic energy of an n-good production system is defined analogously as in physics, see e.g. [16] p. 71. Suppose force $\boldsymbol{F}$ is acting upon production during displacement $d\boldsymbol{Q}$ in the space of accumulated productions of a firm. The instantaneous velocity vector of accumulated production is $d\boldsymbol{Q}/dt = \boldsymbol{q} = (q_1, \ldots, q_n)$ and the momentum vector is $\boldsymbol{p} = (m_1 q_1, \ldots, m_n q_n)$, where the inertial "masses" m_i have units $€ \times y^2/unit_i^2$, $i = 1, \ldots, n$; $unit_i$ is the volume unit for good i and y the unit of time. The inertial mass of production represents all factors restricting changes in the flow of production, such as rigid technology, frictions in adjusting production factors etc. This way all terms $m_i q_i^2, \forall i,$ have unit €.

Now, assuming the inertial masses to be constant, the Newtonian equation $\boldsymbol{F} = d\boldsymbol{p}/dt$ to hold and noticing that $d\boldsymbol{Q} = \boldsymbol{q}dt$, we can write

$$F \cdot d\boldsymbol{Q} = \frac{d\boldsymbol{p}}{dt} \cdot d\boldsymbol{Q} = d\boldsymbol{p} \cdot \boldsymbol{q}.$$

Then, because $d\boldsymbol{p} = (m_1 dq_1, \ldots, m_n dq_n)$ we get

$$d\boldsymbol{p} \cdot \boldsymbol{q} = m_1 q_1 dq_1 + \cdots + m_n q_n dq_n.$$

Integrating this formula we get the kinetic energy of the production system as

$$T = \int \boldsymbol{F} \cdot d\boldsymbol{Q} = \int d\boldsymbol{p} \cdot \boldsymbol{q} = \frac{1}{2} \sum_{i=1}^{n} m_i q_i^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{Q}_i^2. \qquad (1)$$

The kinetic energy of production of good i is thus

$$T_i = \frac{1}{2} m_i \dot{Q}_i^2, \tag{2}$$

where $\dot{Q}_i$ $(unit_i/y)$ is the flow of accumulated production of good i and m_i its inertia (mass) of production with unit $(€ \times y^2)/unit_i{}^2$.

2.2.2. Potential energy of production

In physics, potential energy is stored energy so that even a particle that does not move can have potential energy. If the constraining forces cease to act on the particle, it starts to move in the direction where its potential energy diminishes in the fastest possible way[4]. Potential energy is thus something a particle is "willing" to change to kinetic energy (or motion). Thus we make the following definition: *the potential energy of a firm in the production of good i is its ability to increase its profitability in this production*. If a firm cannot increase its profitability, it has zero potential energy in this production and the firm has no motivation to change its flow of accumulated production of the good.

The idea of economic potential energy is that production is made to earn money. If in the space of accumulated productions there is a direction the firm can enter so that its speed of accumulation of wealth increases, the firm has potential productive (energy). Traditionally, it has been thought in economics that a firm has production potential if it does not operate at full capacity. The above definition changes this thinking so that production potential is only that part of existing capacity that can be exploited in a profitable way. On the other hand, increasing production capacity belongs in the production potential of a firm, too, if this can be made in a profitable way. Adding investments in the current model is, however, a future research problem and is omitted in this study. In other words, if we let a profit-seeking firm operate freely, it starts to move in the direction in the space of its accumulated productions where its wealth accumulates the fastest way. Thus economies and firms with greatest profit opportunities have the greatest production potentials.

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[4] Think of a ball on the top of a hill. It moves down the hill along the path that is steepest in every situation.

Economic kinetic energy originates from the fact that changing the velocity of production of a good needs an effort to overcome the inertial factors resisting these changes. If zero-force is acting upon the production of good *i*, then the velocity of production of good i is constant and profit accumulates in the firm at maximum velocity. In this case the potential energy of the firm with respect to good i is zero, and all energy in this production is in the form of kinetic energy. However, if the force acting upon good i is positive (negative), there is positive (negative) acceleration in production of good *i*, and the kinetic energy of good i increases (decreases) accordingly. If the kinetic energy of production of good i increases, this usually decreases the potential energy (available extra profit) of the firm with respect to good *i*, see section 3.2. However, if increasing returns to scale prevail in production, then the marginal profitability of a good does not decrease when the velocity of production is increased. Thus raising the flow of production of a good with increasing returns to scale creates more energy than it uses, and this production has positive acceleration ad infinity. In the real world, however, this kind of situation exists usually for a limited time. Consider e.g. a mobile phone producer that can expand its production as long as there are regions in the world where no mobile phones exist. Notice that all profitable firms with non-zero production are energetically *perpetual motion machines*; they create more energy than they use, that is, their revenues are greater than their costs. We clarify the defined concepts in the next section.

3. Newtonian and Lagrangian theories of a firm adjusting its flows of production

3.1. Newtonian theory of a firm adjusting its flows of production

Suppose the expected profit function of a two-good firm in competitive markets from time unit *y* is

$$\Pi_e(t) = p_{1e}(t)q_1(t) + p_{2e}(t)q_2(t) - C_e(q_1(t), q_2(t), t), \qquad (3)$$

where $q_i(unit_i/y)$ is the flow of accumulated production of good *i* at time moment *t*, $p_i(€/unit_i)$ the corresponding unit price, $C(q_1, q_2, t)(€/y)$ the cost function of the firm, and $\partial C/\partial q_i(€/unit_i)$ the marginal cost of good

$i, i = 1{,}2$; expected value is denoted by subscript e. Time t in the cost function describes possible effects technical development and changes in input prices may have on the costs with time. This analysis can be extended to n goods, too, but here we operate with only two goods to keep the analysis as simple as possible. Taking the time derivative of Eq. (3) gives:

$$\dot{\Pi}_e = \left(p_{1e} - \frac{\partial C_e}{\partial q_1}\right)\dot{q}_1 + \left(p_{2e} - \frac{\partial C_e}{\partial q_2}\right)\dot{q}_2 + \dot{p}_{1e} q_1 + \dot{p}_{2e} q_2 - \frac{\partial C_e}{\partial t}. \qquad (4)$$

Now, a profit-seeking firm aims to increase its profit with time, i.e. to get $\dot{\Pi}_e > 0$. Because prices are out of control of the firm, and here we do not model the firm's technological development that could make $\partial C_e / \partial t < 0$, the firm can affect its profit only by adjusting its flows of production as

$$m_i \dot{q}_i = p_{ie} - \frac{\partial C_e}{\partial q_i}, \ \ i = 1{,}2, \qquad (5)$$

where $\dot{q}_i = \ddot{Q}_i$ $(unit_i / y^2)$ is the acceleration of accumulated production of good i, $i = 1{,}2$. With positive constant m_i $\left(€ \times y^2 / unit_i{}^2\right)$, Eq. (5) is the simplest possible equation that forces $\dot{q}_i$ and $p_{ie} - \partial C_e / \partial q_i$ to be of equal sign, which makes their product positive. Thus adjusting q_i, $i = 1{,}2$ as in Eq. (5) the firm guarantees that the first two additive terms in Eq. (4) are positive and increase $\dot{\Pi}_e$. Interpreting m_i as the inertial mass of production of good i and $F_i = p_{ie} - \partial C_e / \partial q_i$ as the "force" acting upon the production of good i, Eq. (5) is of identical form as Newton's equation $ma = \boldsymbol{F}$. Assuming that $m_i, i = 1{,}2$ are not infinite or zero, Eq. (5) shows that the firm moves in the space of its accumulated productions in the direction defined by force vector $\boldsymbol{F} = (F_1, F_2)$ where its wealth accumulates the fastest way. The neoclassical framework is a special case of Eq. (5) with $m_i = 0, i = 1{,}2$, i.e. in that case an infinite speed of adjustment is assumed. Now, if $p_{1e} > \partial C_e / \partial q_1$ and $p_{2e} < \partial C_e / \partial q_2$, then $\dot{q}_1 > 0$ and $\dot{q}_2 < 0$, and vice versa. Notice that Eq. (5) corresponds to one of the intuitive dynamic principles in economics stated in [2] referred earlier now presented in an exact form.

Remark! The force vector in Eq. (5) is internal to the firm, and thus no external force is acting upon the firm's production. Eq. (5) is a mathematical "law" that characterizes the behavior of a profit-seeking firm, similarly as Newtonian equation characterizes the behavior of a particle that “wants to minimize its energy”. Notice that the force in Eq. (5) depends on time via the prices and the cost function. This allows us to model the dynamics of production of a firm that has changing revenues and costs in time. Uncertainties may also exist in the prices and in the cost function, and in the force there may be tax and other parameters by which the government can affect the force. This allows us to model economic policy making by applying control theory as in engineering.

Example 1. Let the cost function of a firm be $C(q_1, q_2) = 10q_1 + 10q_2 + 3q_1^2 + 3q_2^2$, where the flows of production are denoted as $q_i, i = 1,2$. The corresponding prices are assumed constant, $p_1 = 70$ and $p_2 = 100$. The expected profit function, where no uncertainties are assumed, is then:

$$\Pi = 70q_1 + 100q_2 - 10q_1 - 10q_2 - 3q_1^2 - 3q_2^2. \tag{6}$$

The constants in Eq. (6) are assumed to have proper units that make Eq. (6) dimensionally well-defined. The Newtonian equations of motion in Eq. (5) with the profit function in Eq. (6) are

$$\begin{pmatrix} m_1\dot{q}_1 \\ m_2\dot{q}_2 \end{pmatrix} = \begin{pmatrix} 60 - 6q_1 \\ 90 - 6q_2 \end{pmatrix},$$

and they have the solutions:

$$\begin{aligned} q_1(t) &= 10 + A_1 e^{-\frac{6t}{m_1}}, \\ q_2(t) &= 15 + A_2 e^{-\frac{6t}{m_2}}, \end{aligned} \tag{7}$$

where $A_i, i = 1,2$ are the constants of integration. The system is stable and will converge into the neoclassical equilibrium: $q_1^* = 10, q_2^* = 15$. In Fig. 1, two solution paths with $m_1 = m_2 = 2$ and $A_1 = A_2 = -5$ are demonstrated. Notice that according to the profit function in Eq. (6), the firm can adjust both its productions independently because the costs of the goods are not interrelated.

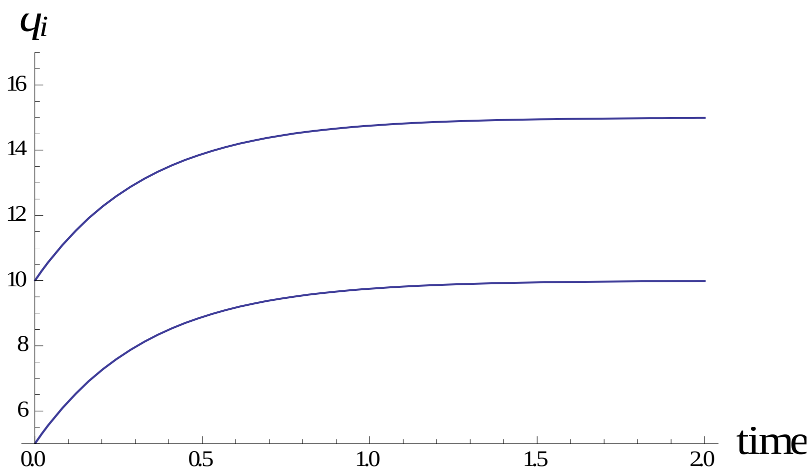

Figure 1. Some solution paths for the flows of production in Eq. (7)

Fig. 2 shows the force field acting upon the production created by the profit function in Eq. (6) and the two singular equations. The force field shows in which direction the profit-seeking firm changes its flows of production in different situations. The length of the vector (arrow) shows the strength of the force field in the corresponding point.

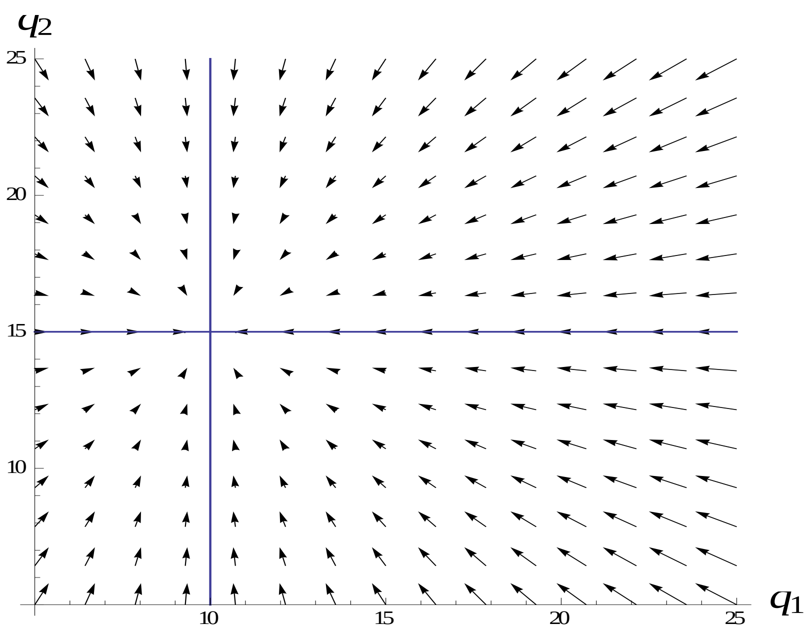

Figure 2. The force field of Eq. (6)
and equations $\dot{q}_1 = 0 \Leftrightarrow q_1 = 10,\ \dot{q}_2 = 0 \Leftrightarrow q_2 = 15.$

The zero-force situation

$$\frac{\partial \Pi}{\partial q_1} = p_1 - \frac{\partial C}{\partial q_1} = 0 \iff q_1 = 10,$$

$$\frac{\partial \Pi}{\partial q_2} = p_2 - \frac{\partial C}{\partial q_2} = 0 \iff q_2 = 15,$$

corresponds to neoclassical theory. In the optimum, the firm cannot increase its profitability and so the firm produces at constant optimal speed. This corresponds to point $q_1 = 10, q_2 = 15$ in Fig. 2.

In the neoclassical theory, firms are assumed to adjust their production after price changes, and so firms are not always assumed to be in their equilibrium state. However, so far the adjustment between two equilibriums has not been modeled formally but only explained verbally. Newtonian Eq. (5) solves this problem by explaining how the firm reaches its new equilibrium after a price change.

Example 2. Let the cost function of a firm be $C = 10q_1 + 10q_2 + 30q_1q_2$, where the last term makes the costs of the two goods interrelated. The corresponding profit function as in Example 1 is

$$\Pi = 70q_1 + 100q_2 - 10q_1 - 10q_2 - 30q_1q_2. \tag{8}$$

The Newtonian equations of production are now

$$\begin{pmatrix} m_1\dot{q}_1 \\ m_2\dot{q}_2 \end{pmatrix} = \begin{pmatrix} 60 - 30q_2 \\ 90 - 30q_1 \end{pmatrix}, \tag{9}$$

and their solutions are:

$$q_1(t) = \frac{1}{2\sqrt{m_1}} e^{-\frac{30t}{\sqrt{m_1m_2}}} \left(6e^{\frac{30t}{\sqrt{m_1m_2}}}\sqrt{m_1} + A_1\sqrt{m_1} + A_2\sqrt{m_2} \right.$$
$$\left. + e^{\frac{60t}{\sqrt{m_1m_2}}}(A_1\sqrt{m_1} - A_2\sqrt{m_2}) \right),$$

$$q_2(t) = \frac{1}{2\sqrt{m_2}} e^{-\frac{30t}{\sqrt{m_1m_2}}} \left(4e^{\frac{30t}{\sqrt{m_1m_2}}}\sqrt{m_2} + A_1\sqrt{m_1} + A_2\sqrt{m_2} \right.$$
$$\left. + + e^{\frac{60t}{\sqrt{m_1m_2}}}(-A_1\sqrt{m_1} + A_2\sqrt{m_2}) \right),$$

where A_1, A_2 are the constants of integration. Setting $m_1 = m_2 = 2$, $A_1 = 10,\ A_2 = 5$, we get the solution paths as shown in Fig. 3.

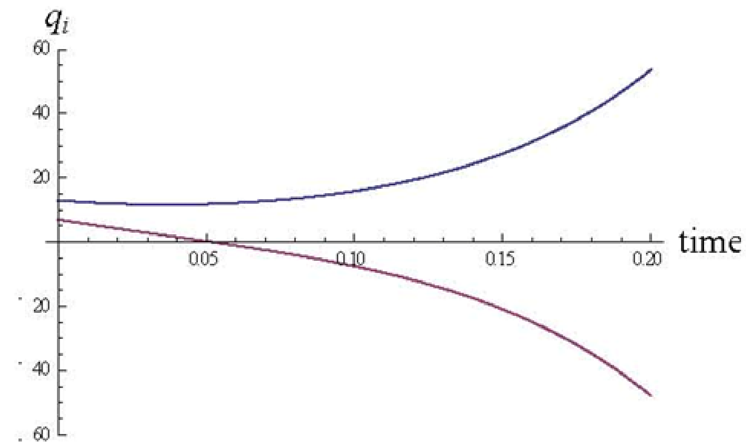

Figure 3. Some solution paths for the flows of production in Eq. (9).

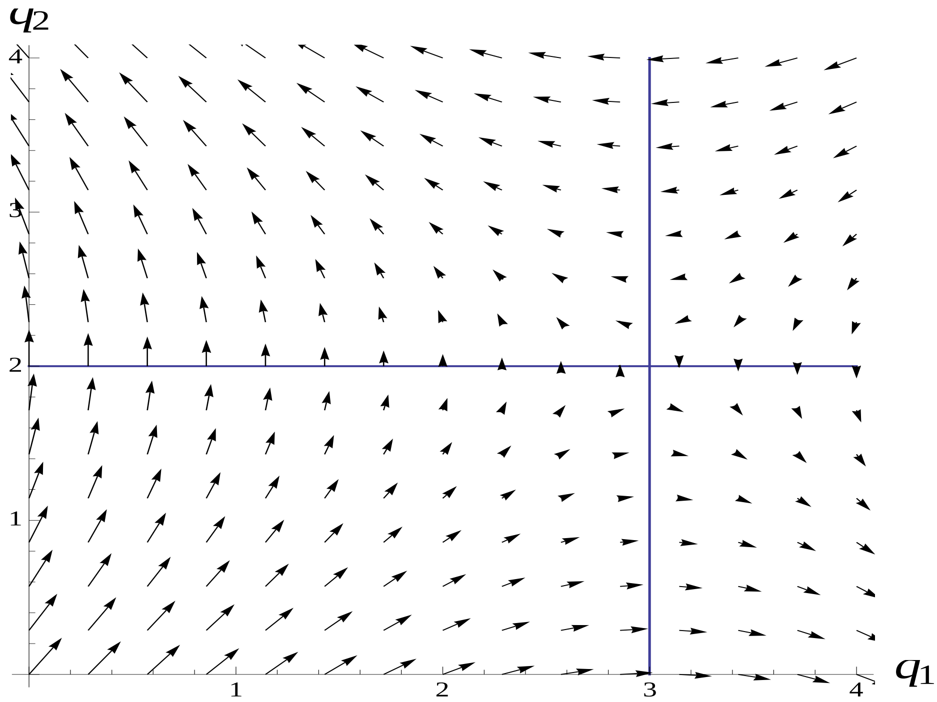

Figure 4. The force field of Eq. (8)
and equations $\dot{q}_1 = 0 \Leftrightarrow q_2 = 2,\ \dot{q}_2 = 0 \Leftrightarrow q_1 = 3$

The force field corresponding to the profit function in Eq. (8) and the two singular equations are shown in Fig. 4. The equilibrium is a saddle and the initial conditions define which one of the two productions expands without limit and which diminishes to zero. This example shows that the production of a good ceases with time if it becomes non-profitable. Notice that this profit function is impossible in the neoclassical framework because no unique optimum exists for this firm.

Example 3. Here we show that time dependent prices and costs can be handled in this framework too. Suppose the prices are $p_1 = 70 + 5\sin(2t)\,, p_2 = 100$, and the cost function of the firm is:

$$C = 10q_1 + 10(1 - 0.5t)q_2 + 3q_1^2 + 3q_2^2.$$

Thus p_1 fluctuates around value 70, p_2 is constant, and the costs of good 2 diminish with time due to e.g. technological progress. The profit function of Eq. (6) takes then the form

$$\Pi = (70{+}5\sin(2t))\, q_1 + 100q_2 - 10q_1 -$$

$$-10(1 - 0.5t)q_2 - 3q_1^2 - 3q_2^2. \tag{10}$$

The corresponding Newtonian equations are

$$\begin{pmatrix} m_1\dot{q}_1 \\ m_2\dot{q}_2 \end{pmatrix} = \begin{pmatrix} 60 + 5\sin(2t) - 6q_1 \\ 90 + 5t - 6q_2 \end{pmatrix}, \tag{11}$$

and their solutions are:

$$q_1(t) = A_1 e^{-\frac{6t}{m_1}} + \frac{20(9 + {m_1}^2) - 5m_1\cos(2t) + 15\sin(2t)}{2(9 + {m_1}^2)},$$

$$q_2(t) = \frac{5}{36}(108 - m_2 + 6t) + A_2 e^{-\frac{6t}{m_2}}.$$

With the initial condition $A_1 = A_2 = 10,\ m_1 = m_2 = 2$, the solutions are graphed in Fig. 5. According to Fig. 5, q_1 oscillates around 10 while q_2 eventually has a positive linear time trend. Permanent growth and cyclical dynamics can thus be explained in the introduced framework, and this example shows that time dependent prices and costs create similar effects on production. This profit function is, however, impossible in the neoclassical framework where time is abstracted away.

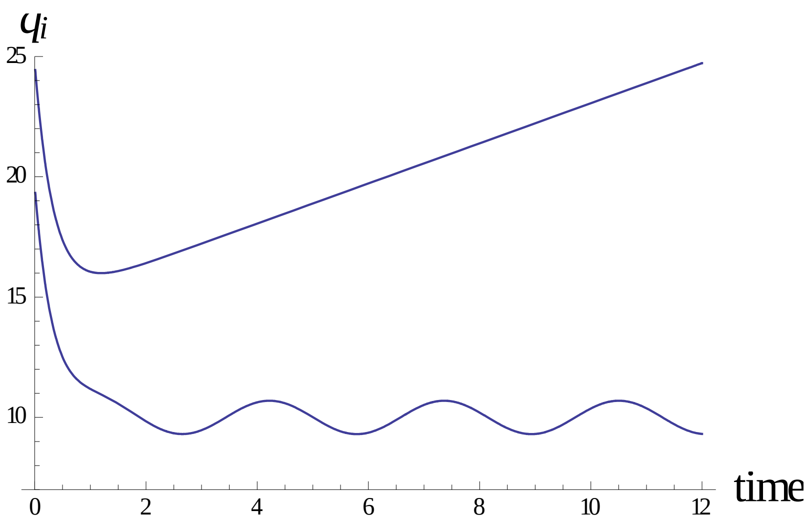

Figure 5. Some solution paths for the flows of production in Eq. (11).

The force field created by the profit function in Eq. (10) is shown in Fig. 6, where time is on the vertical axis, q_1 on the horizontal axis, and q_2 on the third (depth) axis. The figure shows how the force field vanishes in the direction of q_1 in the turning points of q_1.

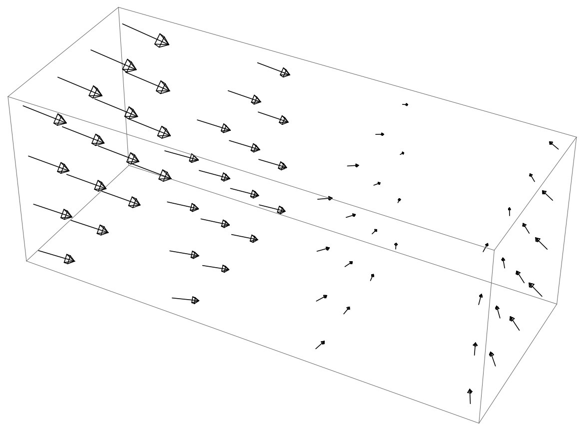

Figure 6. The force field of Eq. (10).

Examples 1-3 show that different profit functions create stable and unstable force fields. A stable force field leads to the profit maximizing state that corresponds to the neoclassical theory. On the other hand, increasing returns to scale, cost advances due to co-production, and time dependent prices or costs may make the force field unstable. Unstable force field leads to the growth or to the collapse of production. These common real world events cannot be explained by using the neoclassical theory.

In [17], the neoclassical theory of production and its assumed duality results are tested by using U.S. manufacturing data. The conclusion is that the theory does not pass the validity tests, and that the primal and the dual theory do not yield similar implications. In [18] is tested the neoclassical theory of production against the Newtonian one presented here by using Finnish data. In every tested 13 industries, Newtonian theory gives reasonable results while the neoclassical one gives a meaningful fit in only one industry, and even that fit is not as good as that given by the Newtonian theory. Thus the neoclassical theory of production has not got support in empirical tests while the Newtonian has.

3.2. Lagrangian theory of a firm adjusting its flows of production

Next we derive Eq. (5) with the profit function in Eq. (3) by using Lagrange's principle. At time moment *t*, the expected extra profit the firm can earn by adjusting its flows of production during the firm's planning time unit (t, t_1) $-$ the potential energy of this production system $P_e(t, t_1)(€)$ $-$ is:

$$P_e(t,t_1) = \int_t^{t_1} \Big(\Pi_e(q_1^*, q_2^*) - \Pi_e\big(q_1(s), q_2(s)\big)\Big)\, ds$$

$$= \int_t^{t_1} \Pi_e(q_1^*, q_2^*)\, ds - \int_t^{t_1} \Pi_e\big(q_1(s), q_2(s)\big)\, ds, \qquad t_1 > t,$$

where q_i^*, *i=1,2*, are the possible fixed flows of production that maximize the expected profit of the firm at time unit (t, t_1), and $q_i(t)$ are current flows. If the optimal flows of production q_i^* do not exist (see Fig. 4), then $\Pi_e(q_1^*, q_2^*)$ is a large enough positive constant expected profit $\Pi_e\big(q_1(s), q_2(s)\big)$ never reaches. The potential economic energy is thus the larger the more the firm can increase its profitability by adjusting its flows of production. At time moment *t*, Lagrange's formalism is then:

$$\underset{Q_i(t),\dot{Q}_i(t)}{\operatorname{Min}} \int L(t)dt\,, \quad L = T_1 + T_2 - P_e(t, t_1)$$

$$= \frac{1}{2} m_1 \dot{Q}_1^2 + \frac{1}{2} m_2 \dot{Q}_2^2 - \int_t^{t_1} \Big(\Pi_e\big(\dot{Q}_1^*, \dot{Q}_2^*\big) - \Pi_e\Big(\dot{Q}_1(s), \dot{Q}_2(s)\Big)\Big)\, ds, \quad (12)$$

where T_i, $i = 1{,}2$, are the kinetic energies of the two goods described in Eq. (2), $\dot{Q}_i(s) = q_i(s)$, *i=1,2,* and L is the Lagrangian function of the production system. The necessary conditions for optimum of the dynamic problem in Eq. (12) with the profit function in Eq. (3) are

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = 0 \iff m_i \ddot{Q}_i = \frac{\partial \Pi_e}{\partial \dot{Q}_i} \iff m_i \dot{q}_i = p_{ie} - \frac{\partial C_e}{\partial \dot{Q}_i}, i = 1{,}2.$$

The Newtonian equations given in Eq. (5) can thus be derived from the Lagrangian function in Eq. (12) by using Lagrange's principle; notice that $\ddot{Q}_i = \dot{q}_i,\ i = 1{,}2$.

4. Newtonian theory of a firm adjusting its accumulated production

In section 3 we modeled the behavior of a firm that produces goods with varying flows of production, and thus the firm can be considered as an adjustor of its flows of production of goods. However, there exist also firms that produce goods only after they have been sold. For example, passenger ships are such goods. Next we model the behavior of a one-good firm that adjusts its accumulated production on the basis of the expected profit from producing an additional unit.

The expected profit from producing one extra unit of the good at time moment *t* is

$$\Pi_{1e}(t) = p(t) - C_{1e}(t),$$

where $p(€/unit)$ is the price at which the firm can sell its product, and C_{1e} $(€/unit)$ the expected unit cost of the good at time moment *t*. The force acting upon the production is then Π_{1e}, and a profit-seeking firm adjusting its accumulated production produces as

$$Q(t) - Q(t-y) = \begin{cases} \dfrac{p(t) - C_{1e}(t)}{m}, & if\ \ p(t) > C_{1e}(t), \\ 0 \qquad\quad , & otherwise, \end{cases}$$

where positive constant $m\,(€/unit^2)$ measures the inertia in this production. This force creates the following time path for accumulated production:

$$Q(t) = Q(t-y) + \frac{p(t) - C_{1e}(t)}{m},$$

and production stops if $\mathrm{p}\,(t) \leq C_{1e}(t)$. Thus accumulated production increases if product price exceeds unit costs, and constant m defines how many units are produced when $\mathrm{p}(t) > C_{1e}(t)$. This example shows that different theories are needed for firms that behave in different ways.

5. Discussion and conclusions

The neoclassical theory of a firm explains constant positive profit maximizing flow of production, and so the theory cannot explain common real world events like business cycles, firms' permanent growth, or bankruptcies. Thus economics needs a more general framework for modeling firms' productions. We introduced here a candidate for such by applying Newtonian and Lagrangian frameworks in physics. These two frameworks were shown to yield the same equations of motion for the production of a profit-seeking firm. Our study is in line with the pioneers of the neoclassical theory that had the idea to define the economic correspondents for the concepts of kinetic and potential energy in physics. We altered the preliminary definitions for these concepts, however, and we hope that these new concepts are useful in the development of similar tools for modeling economic dynamics as has been developed for physics in classical statistical and in quantum mechanics. The introduced framework gives the static neoclassical theory as a special case: the zero-force situation. Thus these two frameworks are consistent.

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