COMPARATIVE ANALYSIS IN STUDY OF CLASSICAL DIFFERENTIAL MAXWELL SYSTEM FOR THE SLOW-GUIDED STRUCTURES

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Abstract. *Two analytical methods in electrodynamics are proposed here basing on the mathematical models in terms of the finite dimensional systems of PDEs (partial differential equations). Both algorithms reduce the original matrix problem to the general wave equation regarding all unknown scalar components of electromagnetic field vector intensities. This wave PDE is solved explicitly in the unified manner irrespectively of concrete boundary value problem statement. Detailed analysis of restrictions in the framework of classical Maxwell theory is suggested as well using both aforesaid analytic techniques in the class of not generalized functions. Efficiency of those research trends is shown in the case of slow-guided structures dealing with electromagnetic wave propagation in the Cartesian coordinate system.*

Keywords and phrases: *classical Maxwell theory, electromagnetic wave propagation, general wave equation, explicit solution.*

1. Introduction

In spite of various numerical methods and subroutines in electrodynamics [1], new analytic procedures remain also required on solution of modern industrial problems dealing with electromagnetic field theory [2]. Rather often, it even happens that desire for more accuracy in approximate computation, in reality gives wrong result having nothing in common with the existing physical or engineering process [1], [2]. Moreover, sometimes the lack of appropriate exact mathematical algorithm leads to severe contingencies of the original problem statement. Thus in [3], the classical differential Maxwell equations for a homogeneous isotropic media in the Cartesian coordinate system were roughly reduced to the degenerate version with vanishing current (charges) $\vec{i} = \vec{i}\,(x, y, z, t)$ and

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charge density $\rho = \rho(x, y, z, t)$. Though the authors in [3] had a claim on the analytic research of electromagnetic wave propagation in the slow-guided structures, their suggested approach not only narrowed mathematical and physical formulation, but also completely spoiled expected results.

It is one of several reasons why the goal of the present paper consists in the presentation of two explicit techniques concerning diagonalization of the finite dimensional square systems of PDEs with piecewise constant coefficients and invertible terms commutative in pairs [4], [5]. Diagonalization means here reduction of the initial matrix problem to the equivalent union of scalar equations where each of them depends on the only one component of the unknown vector field function. Actually, the latter is present implicitly in the aforesaid original system of PDEs.

Returning to the above mentioned set of scalar equations, it is easy to understand that their solving is simpler in comparison with matrix form. Finally, all found solutions uniquely determine the required vector field function. Besides, mathematical modeling of engineering processes using relevant boundary value problems is better to do for scalars avoiding vectors whose study usually remains more complicated and even vague.

Closing this section, it should be noted that since the topic of given article deals with the analytical methods applicable to technical electrodynamics, in further results we are going to show as all virtues of the suggested here investigation, as all drawbacks of the ungrounded trend from [3].

2. Preliminaries

Let the classical differential Maxwell equations in the Cartesian coordinate system and for isotropic homogeneous linear media be given

$$\begin{cases} \ \ \mathbf{rot}\,\vec{H} = \partial_0 \vec{D} + \vec{i}\,, \\ \ \ \mathbf{rot}\,\vec{E} = -\partial_0 \vec{B}, \\ \operatorname{div}\vec{D} = \rho; \ \ \vec{D} = \varepsilon\vec{E}, \\ \operatorname{div}\vec{B} = 0; \ \ \vec{B} = \mu\vec{H}, \\ \vec{i} = \sigma\vec{E}. \end{cases} \tag{1}$$

In (1): $\vec{E}, \vec{H} = \vec{E}, \vec{H}(x, y, z, t)$ are the unknown electromagnetic field vector intensities with scalar components $E_i, H_i = E_i, H_i(x, y, z, t), (i = \overline{1, 3})$; $\vec{D}, \vec{B} = \vec{D}, \vec{B}(x, y, z, t)$ describe the induction of electric and magnetic field respectively; $\vec{i} = \vec{i}\,(x, y, z, t)$ and $\rho = \rho(x, y, z, t)$ determine the current (charges) and charge density; σ, $\mu = \mu_0 > 0$, $\varepsilon = \varepsilon_0 > 0$ denote specific conductivity, relative magnetic and electric permeability of the medium; $\partial_0 = \dfrac{\partial}{\partial t}$; $\quad \text{rot}\, F_k = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_{k1} & F_{k2} & F_{k3} \end{bmatrix}$, $\quad \text{div}\,\vec{F}_k = \sum_{i=1}^{3} \partial_i F_{ki}$ represent the fundamental field operations, where $\partial_1 = \dfrac{\partial}{\partial x}, \partial_2 = \dfrac{\partial}{\partial y}, \partial_3 = \dfrac{\partial}{\partial z}$, and $F_{ki} = F_{ki}(x, y, z, t)\ (k = 1, 2;\ i = \overline{1, 3})$ are the corresponding scalar components of electromagnetic field vector intensities $\vec{F}_1, \vec{F}_2 = \vec{F}_1, \vec{F}_2(x, y, z, t),\ \vec{F}_1, \vec{F}_2 = \vec{E}, \vec{H}$.

Diagonalization of (1) will be done later using two new analytical methods [4], [5]. The first one [4] is the operator generalization of algebraic Gauss method [6]. The second procedure [5] suggests the inverse matrix operator construction. Both of them effectively reduce (1) to the general wave PDE regarding all unknown scalar components of the electromagnetic field vector intensities.

Practically, system (1) was base in [3] investigating electromagnetic wave propagation in the slow-guided structures. Tthough results of [3] aspired to strictly analytical level of study, unfortunately, system (1) was considered in [3] only with hard constraints of $\vec{i} = 0$, $\rho = 0$ and $\sigma = \infty$. Additionally, it was assumed in [3] that time change of electromagnetic field obeyed the law $\exp(i\omega t)$, i. e. $E, H \approx \exp(i\omega t)$, where $i = \sqrt{-1}$ and ω is the vibration frequency.

Perhaps, such breaking of the original physical and mathematical statement was connected with definition of the slow-guided system [3] as the metallic one, not having either magnetics or dielectrics, or charges $\vec{i} = 0$, $\rho = 0$ inside. Perfect conductivity $\sigma = \infty$ and $E, H \approx \exp(i\omega t)$ are also assumed.

The above mentioned restrictions, including influence of ∂_0 on the electromagnetic field vector intensities, naturally take (1) to the following peculiar form

$$\begin{cases} \mathbf{rot}\,\vec{H} = \varepsilon_0 \partial_0 \vec{E} = i\varepsilon_0 \omega \vec{E}, \\ \mathbf{rot}\,\vec{E} = -\mu_0 \partial_0 \vec{H} = -i\mu_0 \omega \vec{H}, \end{cases} \Leftrightarrow \begin{cases} \mathbf{rot}\,\vec{E} + i\mu_0 \omega \vec{H} = 0, \\ -i\varepsilon_0 \omega \vec{E} + \mathbf{rot}\,\vec{H} = 0. \end{cases} \tag{2}$$

Even such cutoff version (2) was considered without longitudinal or transverse waves in [3]. Analytical explicit solution was not also proposed [3].

Nevertheless, the aforesaid methods of [4] and [5] will be shown in the next section for (2) as well, but only when complete exact study of (1) is done.

3. Results

Returning to (1), after obvious transformations, we write this system in the equivalent way

$$\begin{cases} \mathbf{rot}\,\vec{H} = \varepsilon\partial_0\vec{E} + \sigma\vec{E}, \\ \mathbf{rot}\,\vec{E} = -\,\mu\partial_0\vec{H}, \\ \varepsilon\,\mathrm{div}\,\vec{E} = \rho, \\ \mu\,\mathrm{div}\,\vec{H} = 0. \end{cases} \Leftrightarrow \begin{cases} (\varepsilon\partial_0 + \sigma)\vec{E} - \mathbf{rot}\,\vec{H} = 0, \\ \mathbf{rot}\,\vec{E} + \mu\partial_0\vec{H} = 0, \\ ------------ \\ \mathrm{div}\,\vec{E} = \rho/\varepsilon, \\ \mathrm{div}\,\vec{H} = 0. \end{cases} \tag{3}$$

The next step concerns diagonalization of the first pair of equations from (3). The main aim here is obtaining of the general wave PDE regarding all scalar components of electromagnetic field vector intensities.

Since the subsystem of those two equations is homogeneous, method of inverse matrix operator construction [5] as the essential generalization of algebraic approach [6], appears improper. It is well known [6] that homogeneous algebraic square systems have the unique and only zero solution when their determinants equal zero. The last fact completely excludes inverse matrix operator construction whose existence is possible only in the case of nonzero system determinant [5]. That is why diagonalization of the first subsystem from (3) is done by the operator generalization of Gauss method [4]. Namely, application of operators $\mu\partial_0$ and **rot** to the first and the second equations of the mentioned subsystem respectively, after term-by-term addition of those both transformed equations, leads to the equivalent system regarding (3)

$$\begin{cases} (\mathbf{rot}^2 + \mu\partial_0(\varepsilon\partial_0 + \sigma))\vec{E} = 0, \\ \mathbf{rot}\,\vec{E} + \mu\partial_0\vec{H} = 0. \end{cases} \tag{4}$$

The first equation in (4) depends already on the only one electric field vector intensity $\vec{E}$.

The next action of operators $(-\mathrm{rot})$ and $(\mathrm{rot}^2 + \hat{\partial}_0^2)$ on the first and the second equations in (4) with the further term-by-term addition of those two transformed ones, again gives the equivalent system

$$\begin{cases} (\mathrm{rot}^2 + \hat{\partial}_0^2)\vec{E} = 0, \\ \mu\partial_0(\mathrm{rot}^2 + \hat{\partial}_0^2)\vec{H} = 0, \end{cases} \tag{5}$$

where

$$\hat{\partial}_0^2 = \mu\partial_0(\varepsilon\partial_0 + \sigma). \tag{6}$$

Now, the second equation in (5) is dependent on the only one magnetic field vector intensity $\vec{H}$. It is obvious that (5) represents diagonalization of (1) at the electromagnetic field vector level.

Obtaining of the general wave equation regarding all scalar components of $\vec{E}$, $\vec{H}$ demands expressing of (6) in coordinates basing on the identity of the classical field theory [7]

$$\mathbf{rot^2} = \mathbf{grad}\,\mathrm{div} - \Delta, \text{ and } \Delta = \sum_{i=1}^{3} \partial_i^2 \text{ is the Laplace operator.} \tag{7}$$

Owing to the third and fourth equations from (3), use of (7) substantially simplifies (5) giving the following

$$\begin{cases} (\hat{\partial}_0^2 - \Delta)\vec{E} = -\dfrac{1}{\varepsilon}\mathbf{grad}\rho, \\ \mu\partial_0(\hat{\partial}_0^2 - \Delta)\vec{H} = 0. \end{cases} \tag{8}$$

The main assumption here is the specification of scalar function $\rho = \rho(x, y, z, t)$ making $\mathrm{div}\,\vec{E}$ as given.

Influence of the inverse operator $\partial_0^{-1} = \int \mathrm{d}t$ upon the second equation from (8) whose right part is the zero vector, creates again the equivalent system

$$\begin{cases} (\hat{\partial}_0^2 - \Delta)\vec{E} = -\dfrac{1}{\varepsilon}\mathbf{grad}\rho, \\ (\hat{\partial}_0^2 - \Delta)\vec{H} = \dfrac{1}{\mu}\vec{g}(x, y, z). \end{cases} \tag{9}$$

In (9), vector function $\vec{g}(x, y, z)$ is the integration result determined by the physical viewpoint of the concrete engineering problem statement.

It is clear that (9) can be written as the common vector wave PDE regarding field intensities $\vec{E}$ and $\vec{H}$

$$(\hat{\partial}_0^2 - \Delta)\vec{F}_k = \vec{f}_k,\ (k = 1, 2);\ \vec{F}_1 = \vec{E},\ \vec{F}_2 = \vec{H};\ \vec{f}_1 = -\frac{1}{\varepsilon}\mathbf{grad}\rho,$$

$$\vec{f}_2 = \frac{1}{\mu}\vec{g}(x, y, z). \tag{10}$$

Considering (10) in the equivalent coordinate form we have the sought-for general scalar wave PDE

$$(\hat{\partial}_0^2 - \Delta)F_{ki} = f_{ki},\ (k = 1, 2;\ i = \overline{1, 3}), \tag{11}$$

where

$$F_{1i} = \{E_i\}_{i=1}^3,\ F_{2i} = \{H_i\}_{i=1}^3;\ f_{1i} = -\frac{1}{\varepsilon}\partial_i\rho,\ f_{2i} = \frac{1}{\mu}g_i(x, y, z),\ (i = \overline{1, 3});$$

$$\mathbf{grad}\rho = \begin{bmatrix} \partial_1\rho(x, y, z, t) \\ \partial_2\rho(x, y, z, t) \\ \partial_3\rho(x, y, z, t) \end{bmatrix},\ \vec{g} = \begin{bmatrix} g_1(x, y, z) \\ g_2(x, y, z) \\ g_3(x, y, z) \end{bmatrix}. \tag{12}$$

Solution of (11), (12) is done similarly to [8], regardless of boundary conditions and using integral transform method by all spatial variables (x, y, z) [7]. Supported by the technique of [8], the general wave PDE (11), (12) becomes the second order ODE (ordinary differential equation)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\sigma}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\Delta_{tr}}{\mu\varepsilon}\right)_{tr} F_{ki} = {}_{tr}f_{ki}^*,\ (k = 1, 2;\ i = \overline{1, 3}). \tag{13}$$

In (13), all symbols are of the same meaning as in the following formulae from [8]

$$\int_{a_i}^{b_i}(\partial_i^2 F)K_i(x_i, p_i)\,\mathrm{d}x_i = (K_i(\partial_i F) - (\partial_i K_i)F)\,|_{x_i = a_i}^{b_i} + \int_{a_i}^{b_i}(\partial_i^2 K_i)F\,\mathrm{d}x_i =$$

$$= s_i(p_i, x_\nu\ (\nu \neq i;\ \nu = \overline{1, 3}); t) + \eta_i(p_i)_i F_{tr},\ (i = \overline{1, 3}); \tag{14}$$

$$s_i = s_i(p_i, x_\nu\ (\nu \neq i;\ \nu = \overline{1, 3}); t) = (K_i(\partial_i F) - (\partial_i K_i)F)\,|_{x_i = a_i}^{b_i} =$$

$$= s_i(p_i, x_\nu, x_l, t),\ (\nu, l \neq i;\ l > \nu); \tag{15}$$

$$ {}_iF_{tr} = {}_iF_{tr}\ (p_i,\ x_\nu\ \ (\nu \neq i;\ \ \nu = \overline{1,3});\ t) = $$

$$ = \int_{a_i}^{b_i} F(x_i\ (i = \overline{1,3});\ t) K_i(x_i, p_i)\,\mathrm{d}\,x_i = \int_{a_i}^{b_i} FK_i\,\mathrm{d}\,x_i,\ \ (i = \overline{1,3}); \tag{16} $$

$$ F_{tr} = F_{tr}(p_i\ \ (i = \overline{1,3});\ t) = F_{tr}(p, t) = \left(\prod_{\substack{\nu=1 \\ \nu \neq i}}^{3} \int_{a_\nu}^{b_\nu} K_\nu\,\mathrm{d}\,x_\nu \right) {}_iF_{tr} $$

$$ = \int_{a_l}^{b_l} \int_{a_\nu}^{b_\nu} K_\nu(x_\nu, p_\nu) K_l(x_l, p_l)\,{}_iF_{tr}(p_i, x_\nu, x_l, t)\,\mathrm{d}\,x_\nu\,\mathrm{d}\,x_l, $$

$$ p = \bigcup_{i=1}^{3} p_i = (p_1, p_2, p_3),\ (i = \overline{1,3}); \tag{17} $$

$$ \Delta_{tr} = \Delta_{tr}(p) = \sum_{i=1}^{3} \eta_i(p_i);\ \ f^*_{tr} = \tilde{f}_{tr} - \sum_{i=1}^{3} \left(\prod_{\substack{\nu=1 \\ \nu \neq i}}^{3} \int_{a_\nu}^{b_\nu} K_\nu\,\mathrm{d}\,x_\nu \right) s_i, $$

$$ (i = \overline{1,3});\ \tilde{f}_{tr} = \frac{f_{tr}}{(\mu_a \varepsilon_a)^2}, \tag{18} $$

where $x = x_1,\ y = x_2,\ z = x_3;$ the i th integral transform kernel by the argument x_i with parameter p_i is written like that $K_i = K_i(x_i, p_i)$, and the direct integral transform is determined by the expression $S_i = \int_{a_i}^{b_i} K_i(x_i, p_i)\,\mathrm{d}\,x_i$ with the endpoints a_i, b_i of the open integration contour $L_i,\ (i = \overline{1,3})$. Those points can be either finite or infinite real or complex as well [7].

Consideration of integral transforms influence on ΔF by $(x, y, z) = \{x_i,\ (i = \overline{1,3})\}$, allows uniformly presenting the general scalar components of electromagnetic field vector intensities by the expression $F = F(x, y, z, t) = F(x_i,\ (i = \overline{1,3});\ t)$. Then, application of the ith integral transform to $\Delta F = \sum_{i=1}^{3} \partial_i^2 F$ and double integration by parts give (14), (15). In (13), (14) and everywhere below, either the right-hand or the left-hand subscript “tr” means conversion to the corresponding transform. Though in

(15), $i, \nu, l = \overline{1, 3}$, but ν, l take only two values from those three because of two last inequalities in (15). In addition, the second summand $\eta_i(p_i)_i F_{tr}$ from the right side of (14) has the factor $\eta_i(p_i)$ dependent only on the i th integral transform parameter p_i. The mentioned factor is generated by the operation $\partial_i^2 K_i$, $(i = \overline{1, 3})$ in (14). At last, (16) describes the "incomplete" i th transform of F by variable x_i, and derives "complete" transform (17) for F by all spatial arguments $(x, y, z) = (x_i,\ i = \overline{1, 3})$. In (18), conditions for ν, l remain the same as for s_i, $(i = \overline{1, 3})$ from (15), and expression under the sign of $\sum$ in the second formula is similar to the last part of the first equality from (17). Namely,

$$\left(\prod_{\substack{\nu=1\\ \nu\neq i}}^{3} \int_{a_\nu}^{b_\nu} K_\nu \, \mathrm{d}\, x_\nu\right) s_i = \int_{a_l}^{b_l} \int_{a_\nu}^{b_\nu} K_\nu(x_\nu, p_\nu) K_l(x_l, p_l) s_i(p_i, x_\nu, x_l, t)\, \mathrm{d}\, x_\nu \, \mathrm{d}\, x_l,$$

$$(\nu, l \neq i;\ l > \nu;\ i, \nu, l = \overline{1, 3}).$$

Generally, accurate within change of F_{tr}, f_{tr} to ${}_{tr}F_{ki}$, ${}_{tr}f_{ki}$, all symbols in (13) are of the same meaning as in (14)-(18). Inherent difference takes place only in the last expression from (18), where

$$\tilde{f}_{tr} = \frac{f_{tr}}{\varepsilon\mu}.$$

General solution for (13)

$${}_{tr}F_{ki} = {}_{tr}F_{ki}(t, p) = C_1(t, p)\exp(\omega_1 t) + C_2(t, p)\exp(\omega_2 t) \tag{19}$$

is sought by the method of variation of constants [9], where unknown functions $C_j(t) = C_j(t, p)$, $(j = 1, 2)$ represent the solution of system

$$\begin{cases} C_1' \exp(\omega_1 t) + C_2' \exp(\omega_2 t) = 0 \\ C_1'\omega_1 \exp(\omega_1 t) + C_2'\omega_2 \exp(\omega_2 t) = {}_{tr}f_{ki}^*, \end{cases} \quad C_j' = \frac{\partial C_j}{\partial t},\ (j = 1, 2)$$

and look like

$$C_{1,2}(t, p) = \pm \frac{1}{\omega_1 - \omega_2} \int \exp(-\omega_{1,2} t)_{tr} f_{ki}^*(t, p)\, \mathrm{d}\, t + C_{1,2}^*(p). \tag{20}$$

In (20), the unknown functions $C^*_{1,2}(p)$ are found basing on the corresponding transformed initial conditions of the specific boundary value problem. The latter is responsible for the mathematical simulation of the studied physical or engineering process.

It should be noted, that in (19), (20)

$$\omega_{1,2} = \frac{1}{2}\left(-\frac{\sigma}{\varepsilon} \pm \sqrt{D} \right) \tag{21}$$

are the roots of performance (characteristic) equation

$$\omega^2 + \frac{\sigma}{\varepsilon}\omega - \frac{\Delta_{tr}}{\varepsilon\mu} = 0$$

with discriminant

$$D = \left(\frac{\sigma}{\varepsilon}\right)^2 + \frac{4\Delta_{tr}}{\varepsilon\mu}. \tag{22}$$

Substitution of (20) for (19) gives the required general explicit solution of (13)

$$_{tr}F_{ki} = {}_{tr}F_{ki}(t,p) = \sum_{j=1}^{2} C_j(t,p)\exp(\omega_j t) =$$

$$= \sum_{j=1}^{2} \exp(\omega_j t)\left(\frac{(-1)^{j+1}}{\sqrt{D}} \int \exp(-\omega_j t)\,{}_{tr}f_{ki}(t,p)\,\mathrm{d}t + C^*_j(p) \right), \tag{23}$$

where ω_j, $(j = 1, 2)$ and D are determined in (21), (22). Direct check easily confirms that (23) undoubtedly represents the general solution of (13).

The further conversion to the starting wave inverse transform regarding (23) gives function

$$F_{ki} = F_{ki}(x,y,z,t) = \prod_{l=1}^{3} S_l^{-1}\,{}_{tr}F_{ki}\,,\ (k = 1, 2;\ i = \overline{1,3}), \tag{24}$$

where S_l^{-1}, $(l = \overline{1,3})$ are all inverse integral transforms with respect to initially used, and $_{tr}F_{ki}$ is from (23). Hence, explicit expression (24) is the required solution of the general scalar wave PDE (11), (12) and describes all scalar components of electromagnetic field vector intensities. Formulae (23), (24) can be effectively used when specification of the mathematical modeling for engineering problem statement is done and appropriate exact analytic result is demanded.

Since the complete analysis and explicit solution of (1) in the class of not generalized functions are finished, and the first step of this paper's goal is attained, the effective investigation of (2) from [3] can be proposed as well.

System (2) is homogeneous. As it was recently explained for the first subsystem from (3), diagonalization of (2) should be done using operator generalization of algebraic Gauss method [4]. Operator application of **rot** and $(-i\mu_0\omega)$ to the first and second equations of (2) respectively, with the term-by-term addition of both transformed equations, reduces original system to equivalent

$$\begin{cases} (\mathbf{rot}^{\,\mathbf{2}} + i^2\varepsilon_0\mu_0\omega^2)\vec{E} = (\mathbf{rot}^{\,\mathbf{2}} - \varepsilon_0\mu_0\omega^2)\vec{E} = 0, \quad i^2 = (\sqrt{-1})^2, \\ \mathbf{rot}\,\vec{E} + i\mu_0\omega\vec{H} = 0. \end{cases} \tag{25}$$

In (25), the first equation already depends on the only one electric field vector intensity $\vec{E}$. Then, influences of operators (–rot) and $(\mathrm{rot}^{\,2} - \varepsilon_0\mu_0\omega^2)$ on the first and second equations in (25), with further term-by-term addition of both transformed equations, leads again to the equivalent system

$$\begin{cases} (\mathrm{rot}^{\,2} - \varepsilon_0\mu_0\omega^2)\vec{E} = 0, \\ i\mu_0\omega(\mathrm{rot}^{\,2} - \varepsilon_0\mu\ \omega^2)\,\vec{H} = 0, \ i = \sqrt{-1}. \end{cases} \tag{26}$$

In (26), now the second equation depends on the only magnetic field vector intensity $\vec{H}$. Therefore, system (26) closes diagonalization of (2) in terms of vectors. The next stage is writing of (26) in the coordinate (scalar) form using property (7). Obtaining (2), authors of [3] put $\vec{i} = 0, \ \rho = 0$ in the classical version (1). As the result, (7) degenerates into

$$\mathrm{rot}^2 = -\Delta. \tag{27}$$

In its turn, (27) takes (26) to the simplified equivalent system

$$\begin{cases} (-\Delta - \varepsilon_0\mu_0\omega^2)\vec{E} = 0, \\ i\mu_0\omega(-\Delta - \varepsilon_0\mu\ \omega^2)\,\vec{H} = 0, \ i = \sqrt{-1}, \end{cases} \Leftrightarrow \begin{cases} (\Delta + \varepsilon_0\mu_0\omega^2)E_i = 0, \\ (\Delta + \varepsilon_0\mu\ \omega^2)\,H_i = 0, \end{cases} (i = \overline{1, 3}) \tag{28}$$

dependent on all scalar components of electromagnetic field vector intensities. Basing on (28), the required general scalar wave PDE for (2) is written

$$(\Delta + \varepsilon_0\mu_0\omega^2)F_{ki} = 0,\ \ F_{1i} = E_i,\ F_{2i} = H_i,\ (i = \overline{1, 3}). \tag{29}$$

Equation (29) is homogeneous, and partial differential operator in its left part depends only on the spatial variables (x, y, z) not taking into account temporal parameter t, which is natural corollary of the aforesaid ill-founded contingencies [3] on the classical Maxwell system (1). The last fact is the most substantial rough infringement of the original problem statement [3], which not only makes senseless further solution of (29), but in general, deprives (29) any right to be called wave. Even if research of (29) continues, after application of the above mentioned integral transform method [7] by all spatial variables (x, y, z), PDE (29) becomes ordinary algebraic

$$(\Delta_{tr} + \varepsilon_0\mu_0\omega^2)_{tr}F_{ki} = {}_{tr}f^*_{ki},\ \ (k = 1, 2;\ i = \overline{1, 3}). \tag{30}$$

In (30), as in the real general transformed wave equation (13), all symbols are introduced by (14) – (18), and F_{tr}, f_{tr} change into ${}_{tr}F_{ki}$, ${}_{tr}f_{ki}$, where the subscript “tr” implies the earlier function transform by all spatial variables (x, y, z). Here, it should be noted that essential constraint in (30) is connected with (18), where expression for f^*_{tr} has $\tilde{f}_{tr} = 0$. So solution of (30) looks like

$${}_{tr}F_{ki} = \frac{{}_{tr}f^*_{ki}}{\Delta_{tr} + \varepsilon_0\mu_0\omega^2}, \tag{31}$$

and all unknown scalar components of electromagnetic field vector intensities are described by the preceding common formula (24), where the appropriate transforms are given by (31).

Closing analysis of basic results in [3], it is impossible not to engage the authors’ idea concerning solving of (1) at the level of electromagnetic field scalar U and vector $\vec{V}$ potentials. Those functions are introduced by the formulae

$$\vec{E} = -\mathbf{grad}U - \partial_0\vec{V},\ \ \vec{B} = \mathbf{rot}\vec{V} \Leftrightarrow \vec{H} = \frac{1}{\mu}\mathbf{rot}\vec{V}. \tag{32}$$

Further proposed approaches and calculation are done even without aforesaid restrictions [3] for the original Maxwell system (1), which is degenerated to (2) [3]. Hence, basing on (32) let all possible variants of electromagnetic field analytic exact research be considered. The first direction is substitution of (32) for (1) $\equiv$ (3). It gives the equivalent system

$$\begin{cases} (\varepsilon\partial_0+\sigma)\vec{E}-\dfrac{1}{\mu}\mathbf{rot}\,\vec{B}=0, \\ \mathbf{rot}\,\vec{E}+\partial_0\vec{B}=0, \\ ------------ \\ \operatorname{div}\vec{E}=\rho/\varepsilon, \\ \operatorname{div}\vec{B}=0. \end{cases} \Leftrightarrow \begin{cases} \mathbf{rot}(-\mathbf{grad}U-\partial_0\vec{V})+\partial_0\mathbf{rot}\vec{V}=0, \\ (\varepsilon\partial_0+\sigma)(-\mathbf{grad}U-\partial_0\vec{V})-\dfrac{1}{\mu}\mathbf{rot}^2\vec{V}=0, \\ ---------------------- \\ \operatorname{div}(\mathbf{grad}U+\partial_0\vec{V})=-\dfrac{1}{\varepsilon}\rho, \\ \operatorname{div}\mathbf{rot}\vec{V}\equiv 0. \end{cases} \tag{33}$$

Since $\mathbf{rot}\,\mathbf{grad}\,U\equiv\vec{0}$, $\operatorname{div}\mathbf{rot}\vec{V}\equiv 0$, $\operatorname{div}\,\mathbf{grad}U=\Delta$ [7], and operator commutativity in pairs $\partial_0\,\mathbf{rot}=\mathbf{rot}\,\partial_0$, $\partial_0\operatorname{div}=\operatorname{div}\partial_0$ is clear, owing to (6), (7) system (33) becomes its own equivalent

$$\begin{cases} \Delta U+\partial_0\operatorname{div}\vec{V}=-\dfrac{1}{\varepsilon}\rho, \\ \mu(\varepsilon\partial_0+\sigma)\mathbf{grad}U+(\widehat{\partial}_0^2+\mathbf{grad}\operatorname{div}-\Delta)\vec{V}=0. \end{cases} \tag{34}$$

It is easy to find that not all operators in (34) commute in pairs. That's why diagonalization of (34) and its proven reduction to the general wave PDE regarding electromagnetic field potentials is unattainable.

The next variant deals with the direct substitution of (32) for the final diagonalization result of (1), – system (9)

$$\begin{cases} (\widehat{\partial}_0^2-\Delta)\mathbf{grad}U+(\widehat{\partial}_0^2-\Delta)\partial_0\vec{V}=\dfrac{1}{\varepsilon}\mathbf{grad}\rho, \\ (\widehat{\partial}_0^2-\Delta)\mathbf{rot}\vec{V}=\vec{g}(x,y,z). \end{cases} \tag{35}$$

It is obvious that the second equation in (35) depends now on the only one vector potential $-\vec{V}$. It is left now obtaining equation dependent on the only scalar potential U. Application of $\mathbf{rot}$ on the first correspondence in (35), and of $(-\partial_0)$, – on the second one, after the term-by-term addition of both transformed equations basing on $\partial_0\,\mathbf{rot}=\mathbf{rot}\,\partial_0$, leads to the equivalent system

$$\begin{cases} (\widehat{\partial}_0^2-\Delta)\mathbf{rot}\vec{V}=\vec{g}(x,y,z), \\ (\widehat{\partial}_0^2-\Delta)\mathbf{rot}\,\mathbf{grad}U=\dfrac{1}{\varepsilon}\mathbf{rot}\,\mathbf{grad}\rho-\partial_0\vec{g}. \end{cases} \tag{36}$$

It is easy to notice, that both parts of the second equation in (36) are identical zeroes because of $\mathbf{rot}\,\mathbf{grad}U \equiv \vec{0}$, $\mathbf{rot}\,\mathbf{grad}\rho \equiv \vec{0}$ [7] and $\partial_0 \vec{g}(x, y, z) \equiv \vec{0}$. Therefore, (36) consists now of only one equation

$$(\partial_0^2 - \Delta)\mathbf{rot}\vec{V} = \vec{g}(x, y, z), \tag{37}$$

whose structure is more complicated in comparison with the general wave PDE (10)-(13). Even with the given electric field intensity $\vec{E}$ in (32), the search of unknown scalar potential U from (32) with additional condition (37), appears unattainable in spite of detailed study of (37) in scalar form

$$(\partial_0^2 - \Delta)\begin{bmatrix} \partial_2 V_3 - \partial_3 V_2 \\ \partial_3 V_1 - \partial_1 V_3 \\ \partial_1 V_2 - \partial_2 V_1 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \tag{38}$$

and further complete diagonalization of (38). Since (38) is inhomogeneous, the indicated procedure can be done by the inverse matrix operator construction [5]. Remembering that uniqueness of solution for inhomogeneous algebraic system exists only with its nonzero determinant [6], we base on the operator generalization of mentioned method [5] and compute the determinant for (38):

$$\det(38) = \det\left((\partial_0^2 - \Delta)\begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \right) \equiv 0.$$

The last expression proves the lack of unique solution for (38) and therefore, impossibility of potentials' $U, \vec{V}$ explicit construction.

The last opportunity of exact retrieval of unknown potentials $U, \vec{V}$ remains. It implies study of (32) as corresponding system with respect to unknown $U, \vec{V}$ and given electromagnetic field vector intensities $\vec{E}, \vec{H}$

$$\begin{cases} \mathbf{rot}\vec{V} = \mu\vec{H}, \\ \mathbf{grad}U + \partial_0 \vec{V} = -\vec{E}. \end{cases} \tag{39}$$

System (39) is inhomogeneous because $\vec{E}, \vec{H}$ are known here. Hence, as in the preceding case, (39) has the unique solution with its nonzero determinant. Unfortunately, even for (39), the explicit seek of the field potentials $U, \vec{V}$ is impossible because determinant of this system is identical zero

$$\det(39) = \det\begin{bmatrix} 0 & \mathbf{rot} \\ \mathbf{grad} & \partial_0 \end{bmatrix} \equiv 0,$$

basing on $\mathrm{rot}\ \mathrm{grad} \equiv \vec{0}$ [7].

Analysis of all obtained here results confirms appropriateness and efficiency of those two proposed exact methods [4], [5] used for explicit electromagnetic field study. Ungrounded restrictions concerning "rough" physical / mathematical simplification of the original fundamental statement break connection between investigations of real engineering phenomena and correct analytic solution of corresponding applied problems [3].

Finally, further support of procedures [4], [5] evokes comparison of [5] with its application to (1) when the charge density $\rho = \rho(x, y, z, t)$ is a fortiori indeterminate. Then the first equation from (5) basing on (7) can be written in scalar form

$$\begin{cases} \widehat{A}_{23}E_1 + B_{12}E_2 + B_{13}E_3 = 0, \\ B_{12}E_1 + \widehat{A}_{13}E_2 + B_{23}E_3 = 0, \\ B_{13}E_1 + B_{23}E_2 + \widehat{A}_{12}E_3 = 0. \end{cases} \tag{40}$$

System (40) looks almost like analogous result in [5], accurate within zero right sides of equations and change of operators $\tilde{A}_{ij}$ into

$$\widehat{A}_{ij} = \widehat{\partial}_0^2 - \Delta + \partial_i^2, \ (i = 1,\ 2;\ j = 2,\ 3;\ i \neq j), \tag{41}$$

where $\widehat{\partial}_0^2$ is denoted by (6). Operators $B_{ij} = \partial_i \partial_j$, $(i = 1,\ 2;\ j = 2,\ 3;\ i \neq j)$ are here the same as in [5].

Application of unifying technique [4] to homogeneous operator systems, takes (41) to the general wave PDE with respect to all scalar components of electric field vector intensity $\vec{E}$

$$\widehat{\partial}_0^2(\widehat{\partial}_0^2 - \Delta)E_i = 0 \quad (i = \overline{1,\ 3}). \tag{42}$$

Solution of (42) is done similarly to (10) using the integral transform method [7]. Introduction of new function

$$\Phi_i = \Phi_i(x, y, z, t) = \widehat{\partial}_0^2 E_i, \ \ (i = \overline{1,\ 3}) \tag{43}$$

reduces (42) to the following second-order equation

$$(\widehat{\partial}_0^2 - \Delta)\Phi_i = 0 \quad (i = \overline{1, 3}). \tag{44}$$

Instead of the original fourth-order one, we get (44) of less order. Then, in exactly the same way as in [8], at first (44) is solved and function $\Phi_i \ (i = \overline{1, 3})$ is found. After that, (43) is considered as the particular case of (44) with $\Delta \equiv 0,$ but with already given nonvanishing function $\Phi_i \ (i = \overline{1, 3})$ and unknown $E_i \ (i = \overline{1, 3}).$ Analytical study of (43), identically to [8], gives all scalar components of the desired electric field vector intensity $\vec{E}$. Here, the difference consists only in $\widehat{\partial}_0^2$ from (6) and operators $\widetilde{\partial}_0^2 = \mu_a \varepsilon_a (\partial_0^*)^2 + (\sigma\mu_a + r\varepsilon_a)\partial_0^* + r\sigma,\ \partial_0^* = \partial_0 \pm \lambda$ in [8].

By analogy to (42), the general wave PDE for $\vec{H}$ is got and solved also explicitly.

4. Conclusions

Closing the article, it should be noted that the assigned task is completely fulfilled.

Actually, the suggested approach uniting those two analytic operator diagonalization methods can be applied to any type of the finite dimensional square system of operator equations with piecewise constant coefficients and invertible terms commutative in pairs.

Both proposed here methods act in the class of not generalized functions and irrespectively of the specific boundary value problem statement. Those two virtues simplify as analysis of the studied engineering or physical process, as its mathematical modeling and further explicit solution.

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