FROM MAXWELLIAN ELASTIC COLLISIONS TO INTIMATELY CONNECTED KINETIC WEALTH EXCHANGE MODELS

Maximilien L. KÜRTEN* and Karl E. KÜRTEN**

Abstract. *Kinetic wealth-exchange models and molecular dynamical models of particles which exchange energy during their collisions are formally intimately connected. Assuming that the density distribution of the velocities is identical in all regions of space we ignore the spatial positions of the individual particles and let randomly selected pairs of particles collide in contrast to the well established recipes used in molecular dynamics theory. Instead, we simply select randomly pairs of particles, let them virtually collide and mutually exchange a fraction of their energies.*

In spite of this dramatic simplification the velocity as well as the resulting energy distributions eventually compare well with the theoretical Maxwell-Boltzmann distributions. We further explore models uniquely based on one-dimensional kinetic exchange rules, where in addition, the d-dimensional velocity space has been neglected. Since the dimension d is incorporated in the physical interactions taken from collisions in the d-dimensional velocity space of the first part of the paper we recover excellently all the physical quantities.

Keywords: *Maxwell-Boltzmann distribution, energy exchange, wealth exchange, gamma distribution.*

1. Introduction

Kinetic exchange models are stochastic models which can be straightforwardly adapted to study problems in a variety as disciplines, such as economics and social sciences [1, 2, 3, 4, 5, 6]. The Maxwell Boltzmann velocity distribution applies to ideal gases, where the particles do not constantly interact with each other but move freely between short collisions, where the only interactions taken into account are binary collisions. It describes the probability of a particle's velocity, its momentum or energy as a function of the mass and the temperature.

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* Ecole Supérieure d'Électricité, 91192 Gif-sur-Yvette Cedex, France

** Faculty of Physics, Vienna University, A-1090 Vienna, Austria. Department of Physics, Loughborough University, LE11 3TU, UK.

Provided that the probability distribution is assumed to be independent of the position of the particles, we have the homogeneous Boltzmann equation Let us first treat particles which are all equal and indistinguishable with equal masses.

2. Boltzmann distribution

Note that there are a variety of ways to derive the Maxwell-Boltzmann statistics. Boltzmann theory assumes that the d components of the velocity $(v = v_1, v_2, ..., v_d)$ are normally distributed with mean zero and variance σ^2, where the variance is usually identified with the temperature T.

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} . \quad (1)$$

Note that the variance σ^2 is proportional to the total energy of the system. The velocity distribution $P(v)$ can be considered as the product of d independent normally distributed variables together with the appropriate d-dimensional volume element, which for dimension three is $4\pi v^2$. The resulting velocity and energy distributions can be immediately obtained by standard transformation techniques $\left(v \to E = \frac{1}{2}v^2\right)$. We have the well known result

$$P(v) = \frac{v^{d-1}}{\Gamma\left(\frac{d}{2}\right) 2^{\frac{d}{2}-1} \sigma^d} e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^2} \quad (2)$$

with mean $<v> = \sqrt{2}\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)}\sigma$. The corresponding energy distribution is the well known gamma distribution

$$P(E) = \frac{E^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)\sigma^d} e^{-\frac{E}{\sigma^2}} \quad (3)$$

with mean $< E > = \frac{d}{2}\sigma^2$. Note that for $\sigma^2 = 1$ we have the chi-square distribution. Figure 1 depicts the density distributions $P(x), P(v)$, and $P(E)$ for dimension $d = 7$ (from left to right).

3. Space-independent elastic collision dynamics

Let us now assume we have a system containing N hard disks in d-dimensional velocity-space. The particles are not set up on a lattice space, since r-space is irrelevant in our simulation. There is no free flight path between the virtual collisions. A loop over all N particles involves N collisions, i.e. each particle collides twice on average during one loop. The initial velocities are chosen at random according to a specific density distribution with variance σ^2 and the total momentum, which is a conserved quantity, and equals zero. We repeatedly choose a pair of particles (i, j) at random which is supposed to "virtually" collide. The post-collisional velocities $\vec{v}'_1$ and $\vec{v}'_2$ are adjusted via

$$\vec{v}'_i = \vec{v}_i + \vec{\Delta} \qquad \vec{v}'_j = \vec{v}_j + \vec{\Delta} \tag{4}$$

with the momentum transfer vector

$$\vec{\Delta} = \gamma \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|} \tag{5}$$

where $\vec{r}_{ij}$ is a vector still to be specified representing the "virtual" distance vector point from the center of particle i to the center of particle j. v_i and v_j

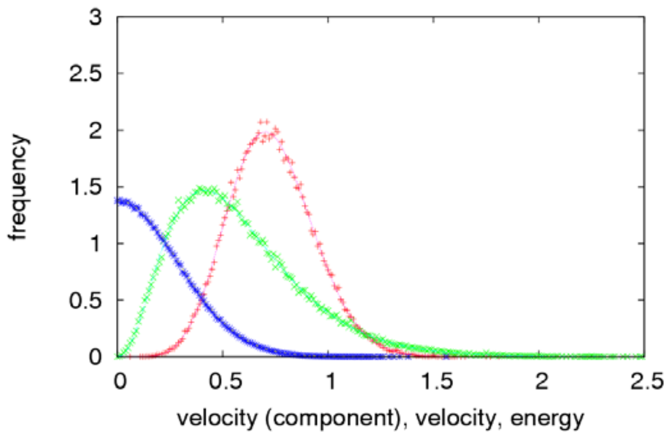

Figure 1. Distribution of the velocity components, velocities and energies after 10 collisions per particle on average (from left to right) for dimension $d = 7$.

are the velocities of particle *i* and particle *j* who are located at the virtual $r-$ space at position r_i and position r_j respectively. The factor γ has to be chosen such that the sum of the post-collisional energies E_i' and E_j'

$$E_i' = \frac{1}{2}(\vec{v}_i + \vec{\Delta})^2 \qquad E_j' = \frac{1}{2}(\vec{v}_j + \vec{\Delta})^2 \tag{6}$$

is conserved. The factor γ then necessarily takes the form:

$$\gamma = \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|}\vec{v}_{ij} = \cos(\alpha_{ij})\,|\vec{v}_{ij}| \tag{7}$$

with the difference vector $\vec{v}_{ij} = \vec{v}_j - \vec{v}_i$. Inserting the cosines of the angles $\alpha_1 = \sphericalangle(\vec{v}_1, \vec{r}_{12})$ and $\alpha_2 = \sphericalangle(\vec{v}_1, \vec{r}_{12})$ between $\vec{r}_{12}$ and the velocity vectors $\vec{v}_i$ and $\vec{v}_j$ by:

$$\cos\alpha_i = \frac{\vec{v}_i \cdot \vec{r}_{ij}}{|\vec{v}_i||\vec{r}_{ij}|} \quad \cos\alpha_j = \frac{\vec{v}_j \cdot \vec{r}_{ij}}{|\vec{v}_j||\vec{r}_{ij}|} \tag{8}$$

into the energy Eq.(8) we find

$$E_i' = E_i - E_i\cos^2\alpha_i + E_j\cos^2\alpha_j \qquad E_j' = E_j - E_j\cos^2\alpha_j + E_i\cos^2\alpha_i. \tag{9}$$

Note that particle *i* looses a fraction of its own energy E_i specified by $\cos^2\alpha_i$ while particle *j* gains a fraction of the energy E_j of its counterpart specified by $\cos^2\alpha_j$. Starting from a uniform distribution of the velocity components $v_i \in [-a,+a]$ with $\sigma^2 = \frac{a^2}{3}$ and a normal distribution for the components of the virtual distance vector $\vec{r}_{ij}$ (Eq.(5)), we eventually end up with the expected Mawell Boltzmann distributions with the correct dimension *d*. After sufficiently many collisions the distribution of the velocity components converges to the normal distribution, where the variance σ^2 prescribed in the initial condition, is preserved. In general, it is important that the components of the distance vectors $\vec{r}_{ij}$ have to be chosen "sufficiently" random such that the central limited theorem can be applied. Otherwise, the distributions will often still be Maxwell Boltzmann, however there might arise problems to reproduce the correct dimension *d*. Figure 1 depicts density distributions for the velocity components, the velocities and the energies after 10 collisions per particle on average. They are indistinguishable from the analytic Maxwell Boltzmann prescription given in Eqs. (1-3).

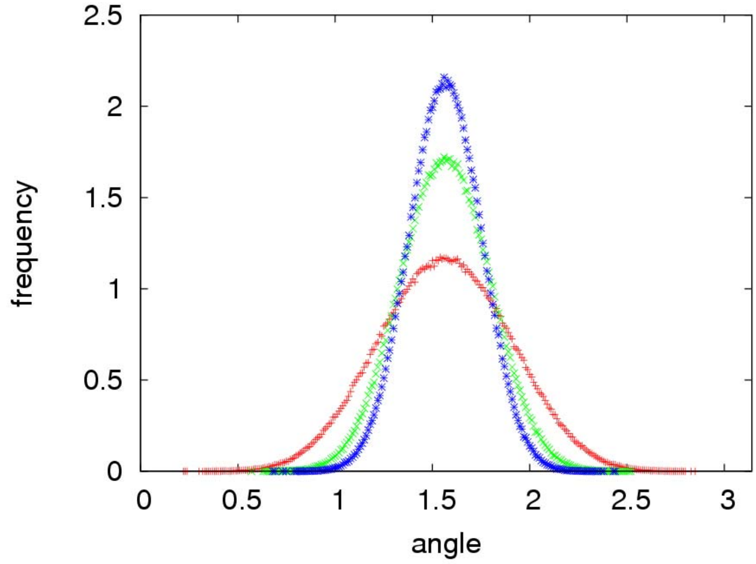

Figure 2. Angle distributions of $\sphericalangle(\vec{r}_{ij}, \vec{v}_{ij})$ for dimension $d = 10$, $d = 20$, and $d = 30$ (from right to left).

Let us now have a look at various angles specified by standard scalar products of two unit vectors: $\cos\alpha_i = \frac{\vec{v}_i \cdot \vec{r}_{ij}}{|\vec{v}_i||\vec{r}_{ij}|}$, $\cos\alpha_j = \frac{\vec{v}_j \cdot \vec{r}_{ij}}{|\vec{v}_j||\vec{r}_{ij}|}$, $\cos\alpha_{ij} = \frac{\vec{v}_{ij} \cdot \vec{r}_{ij}}{|\vec{v}_{ij}||\vec{r}_{ij}|}$, and $\cos\alpha_{ii'} = \frac{\vec{v}'_i \cdot \vec{r}_i}{|\vec{v}'_i||\vec{r}_i|}$ which appear in our simulations and attract our interest, since they are measurable physical quantities. During the simulations all scalar products could be identified as normally distributed with mean zero and variance $\sigma^2 = \frac{1}{d}$. In principle, this is only to be expected for the collision angle $\sphericalangle(\vec{v}_i, \vec{r}_j)$, since due to the molecular chaos concept the relative velocities of the particles are less and less correlated with increasing dimension d. In contrast, in any scalar product where the distance vector r_{ij} explicitly appears, we have strong correlations between the distance vector r_{ij} and the velocity vectors $\vec{v}_i, \vec{v}_j, \vec{v}_{ij}$. These random unit vectors are **NOT** independent and hence do not have a scalar product with mean zero. This physical anomaly that all appearing angles are centered around the value $\frac{\pi}{2}$, can be well explained. It is well known that the density distribution of a scalar product between two random unit vectors in d-dimensions is of the form [7]:

$$P(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (1-x^2)^{\frac{d-2}{2}} \qquad x \in [-1,1]. \tag{10}$$

However the above theorem only holds as long as the two unit vectors are independent. The distribution is exact for $d = 1$ and $d = 2$ and is close to a normal distribution for larger values of d. Moreover, the variance depends on the dimension d and is approximately $\frac{1}{d}$. In accord with our simulations the scalar products are normally distributed around zero and the distributions of the corresponding angles are centered around $\frac{\pi}{2}$, since in our "zero" model we assume that the two random vectors are independent. To summarize: the components of the unit vector $\frac{\vec{r}_{ij}}{|\vec{r}_{ij}|}$ are uniformly distributed on the $(d-1)$ – dimensional hypersphere, however they are strongly correlated with the velocity vectors $\vec{v}_i$ and $\vec{v}_j$ such that the relevant angles show physical anomaly.

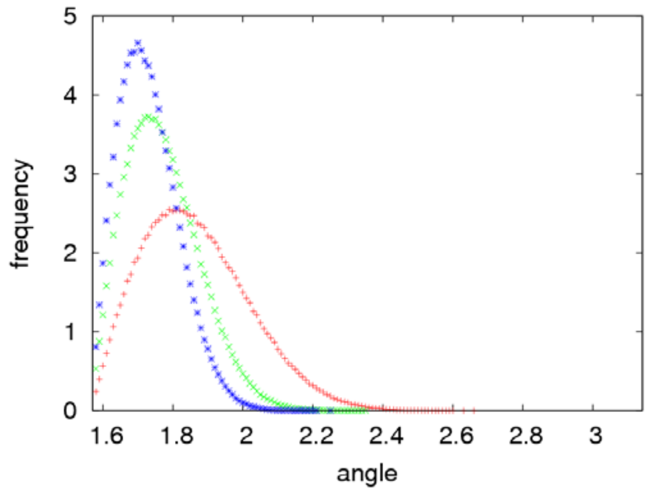

Figure 3. Density distribution of the angles $\sphericalangle(\vec{r}_{ij}, \vec{v}_{ij})$, for dimension $d = 10$, $d = 20$, and $d = 30$ which due to correlations are highly sewed.

4. Improved realistic recipe for virtual collisions

There is one angle α_{ij}, specified by $\cos\alpha_{ij} = \frac{\vec{v}_{ij} \cdot \vec{r}_{ij}}{|\vec{v}_{ij}||\vec{r}_{ij}|}$, where the density distribution can be derived analytically. Making use of the known distribution of the impact factor b, defined as:

$$b = D\sin(\sphericalangle(\vec{r}_{ij}, \vec{v}_{ij})) \tag{11}$$

where D is the diameter of the disks and the probability distribution is of the form:

$$P(b) = \frac{d-1}{D}\left(\frac{b}{D}\right)^{d-2}. \tag{12}$$

Note that the distribution is only uniform for the dimension d = 2. One can show further [8] that the angle $a_{ij} = \sphericalangle(\vec{r}_{ij}, \vec{v}_{ij})$ which appears explicitly in the adjustments of the velocity vectors, $\vec{v}_i$ and $\vec{v}_j$ Eq. (5), follows the density Distribution

$$P(\alpha_{ij}) = \begin{cases} -(d-1)\sin\alpha_{ij}^{\,d-2}\cos\alpha_{ij} & \text{if } \alpha_{ij} \in \left[\frac{\pi}{2}, \pi\right]. \\ 0 & \textit{otherwise} \end{cases} \tag{13}$$

Figure 3 shows that this angle is not at all centered around $\frac{\pi}{2}$, but restricted to the interval $\left[\frac{\pi}{2}, \pi\right]$. The maximum value is reached at $\alpha_{ij} = \pi - \arctan\sqrt{d-2}$.

Figure 3 depicts that with increasing dimension d the distribution gets more and more peaked and approaches the value $\frac{\pi}{2}$ eventually becoming singular for $d \to \infty$. The shape if $P(\alpha_{ij})$ reflects the fact that the vectors $\vec{r}_{ij}$ and $\vec{v}_{ij}$ are not at all independent and consequently the mean of the corresponding scalar product does not take the value zero. In order to stay with physical interactions the improved recipe for the virtual collision process is now as follows:

1) Choose the components of $\vec{r}_{ij}$ according to a normal distribution with $\sigma^2 = 1$.

2) Calculate $\cos(\alpha_{ij}) = \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|} \frac{\vec{v}_{ij}}{|\vec{v}_{ij}|}$.

3) if $\alpha_{ij} \notin \left[\frac{\pi}{2}, \pi\right]$ go back to 1), otherwise accept $\vec{r}_{ij}$ with probability

$$P(\alpha_{ij}) = -(d-1) \sin \alpha_{ij}{}^{d-2} \cos \alpha_{ij} . \tag{14}$$

Indeed, this physical constraint yields angle distributions which compare well with molecular dynamical simulations [8].

5. Energy versus Wealth-Exchange

It is meanwhile widely believed that in certain aspects wealth exchange models composed of an assembly of N indistinguishable agents, each of which has a certain wealth $E_i, (i = 1,...N)$, can be treated with the tools of the Boltzmann legacy [9] even if the interaction rules might not be that physical. Moreover, interaction rules which meet the physics will usually lead to Maxwell Boltzmann distributions. The hard spheres now correspond to the agents, the energies are identified with the wealth of the agents, while the binary collisions correspond to wealth-exchange interactions between two the agents. The interactions occur pairwise between randomly chosen agents who exchange wealth or energy according to the following energy exchange rule

$$E_i' = E_i - E_i \cos^2 \alpha_i + E_j \cos^2 \alpha_j$$

$$E_j' = E_j - E_j \cos^2 \alpha_j + E_i \cos^2 \alpha_i . \tag{15}$$

Note that these equations (Eq. (12)) taken from the previous chapter describe the energy transfer of hard spheres. However, in contrast to wealth exchange models, in the hard spheres problem the two angles α_i and α_j are specified by the physics of collisions Eq. (2) and are highly correlated with the distance vector $\vec{r}_{ij}$. According to Eqs. (13) and (14) they depend on the two velocity vectors $\vec{v}_i$ and $\vec{v}_j$ as well as on the angle $\sphericalangle(\vec{r}_{ij}, \vec{v}_{ij})$ involved in the impact factor *b*. In contrast to the physics of collision, wealth exchange models usually assume that the decisive angles α_i and α_j are independent, usually drawn from rather specific probability distributions according to the needs and aims of the specific model. Different wealth exchange models differ in their recipes of specifying the distribution of the angles α_i and α_j (Eq.(15)) [9]. The distributions obviously depend crucially on the particular choice of the decisive angles stemming from *d*-dimensional space.

5.1. Totally random energy exchange

One of the simplest wealth exchange model is characterized by a completely random exchange of energy somewhat reminiscent of the interaction rules of an ideal gas. One assumes that during each collision the sum of the energy of the two agents E_i and E_j randomly splits between the two agents with equal probability to gain or to lose energy. This version corresponds to the choice $\cos^2\alpha_i = 1-\cos^2\alpha_j$, where either α_i or α_j are chosen at random according to a specific probability distribution. In terms of the energy transfer we would have

$$E_i' = E_i \sin^2\alpha_i + E_j \cos^2\alpha_j$$

$$E_j' = E_j \sin^2\alpha_j + E_i \cos^2\alpha_i \tag{16}$$

with the choice $\sin^2\alpha_i = \cos^2\alpha_j$. Iterating Eq.(16) eventually yields the well-known Maxwell-Boltzmann distribution for the energy for dimension $d = 2$.

In another, somewhat more realistic model it is always the energetically richer agent who transfers a fraction of his energy to energetically poorer agent. A possible description for the energy transfer is straightforward. Assuming that $E_i < E_j$ we have:

$$\cos^2\alpha_i > \frac{1}{1+\dfrac{E_j}{E_i}}. \tag{17}$$

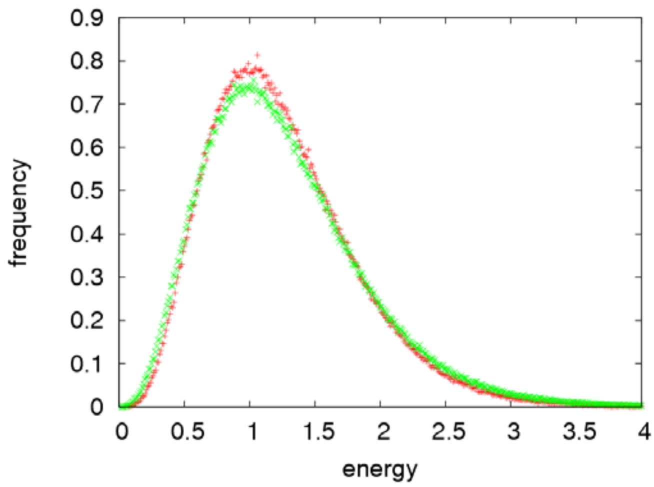

Figure 4. Energy distributions for *d*-dimensional velocity space and one dimensional energy space.

During the collision process the factor $\cos^2 \alpha_i$ is simply chosen from the specific interval

$$\cos^2 \alpha_i \in \left[\frac{1}{1 + \frac{E_j}{E_i}}, 1 \right]. \tag{18}$$

Also this model turns out to have an equilibrium distribution given by the familiar Gamma distribution with dimension $d \approx 2.5$.

Last not least we apply our suggested improved collision algorithm presented in chapter 4, where the density distribution of the angles α_{ij} have been explicitly included as a constraint. Making use of these distributions taken from physics, where the angles α_i and α_j are not centered around $\frac{\phi}{2}$ with variance $\sigma^2 = \frac{1}{d}$, which one only finds for scalar products of two independent randomly chosen unit vectors, we find the energy distributions in terms of the gamma distribution rather close to the dimensions embedded in the corresponding scalar products. Figure 4 shows convincingly that also the one-dimensional energy exchange version can compete with the *d*-dimensional version in velocity space, provided that that the corresponding density distributions stemming from the *d*-dimensional velocity space are incorporated in the interactions.

6. Discussion

Due to substantial differences between the collision dynamics of ideal gas particles and the modeling of wealth exchange interactions, only models based rather closely on the kinetic theory of gases predict the familiar Maxwell-Boltzmann distributions including the relevant physical angles for a prescribe dimension *d*. In most models currently practiced, the rules are derived from plausible assumptions in an ad hoc manner, which is clearly in marked contrast to Boltzmann's original theory, where the microscopic collisions are governed by strict physical laws given for example by Eq.(13), where the angles α_{ij}, although they can be considered as random, are strongly governed by correlations. On the other hand, wealth exchange models often aim at a Pareto tail of the probability distribution of wealth [10] which is a manifestation of the existence of very rich agents, a manifestation of an unequal distribution of wealth. Whenever the interactions are governed by more or less exotic interaction rules, the result is the appearance of more or less unphysical and exotic energy distributions reported earlier [9]. They take forms such as uniform distributions, truncated exponential distributions, Gamma distributions, Gaussian distributions, mixed exponential and inverse power law distributions, the celebrated Pareto distributions. Note that most of these density distributions can also be found in the theory of complex biological systems. To conclude, the analogy between the theory of market economics and the kinetic theory of ideal gases can be well exploited provided one remains on the path of physics.

7. Acknowledgements

The author thanks M. Neumann, H. Posch and F. Vessely for various helpful discussions. This work was partly supported by the Royal Society London (International Joint project 2009/R3) and the European Science Foundation network-programme, AQDJJ.

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