THE COMPLETE SET OF THE 33 MODELS AND THEIR ANALYTICAL SOLUTIONS ASSOCIATED WITH A PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER

Nicolae POPOVICIU*

Abstract. *The paper presents a complete set of 33 problems (7 types of problems for the general case – called also the mixed case, 3 types of problems for the reduced case and a lot of modified cases) and all the corresponding solutions associated with a partial differential equation of the second order (PDE2). The set of all these seven problems associated with* ***the general case*** *is denoted by* P_{OON}, P_{ONO}, P_{ONN}, P_{NOO}, P_{NON}, P_{NNO}, P_{NNN} *(or OON etc.) and* ***the reduced case****:* P_{ON}, P_{NO}, P_{NN} *(or ON etc). Sometimes the equation or the attached condition could be modified in comparison with the standard case. The letter m after the letter N means that the non-homogenous equation or condition N has been modified.*

Keywords**:** *Partial differential equation, general case, reduced case, hyperbolic type, first mixed problem, second mixed problem, vibrating rope.*

1. Introduction. Notations

Shortly, a partial differential equation of the second order is denoted by PDE2 and its unknown function is $u(x,t)$, where $x \in R$, $t \in [0,\infty)$; x and t are independent variables. The variable x indicates the place on real axis and the variable t represents the time. The extension for $x \in R^n$ isn't done here.

Here we deal with a *PDE* having the form (1):

$$a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + \alpha(t)\frac{\partial^2 u}{\partial t^2} + \beta(t)\frac{\partial u}{\partial t} + [c(x)+\gamma(t)]u + F(x,t) = 0 \quad (1)$$

A combination of the analytical functions $a, b, c, \alpha, \beta, \gamma$ yields many particular problems. In the technical problems these functions are real constants.

* Hyperion University of Bucharest, Faculty of Mathematics-Informatics, e-mail: nicolae. popoviciu@yahoo.com

The general case. We denote such a problem by P_{XXX}, where X designates an equation or a set of conditions attached to the equation.

Sometimes the equation and/or the condition X is modified in the comparison with the standard case. We denote it by the letter m, where m is lying after X. For example $XmXX$ means that the partial differential equation X is modified and $XXXm$ means that the third condition X is modified.

A problem of type XXX or $XmXX$ is called **the first mixed problem.**

A problem of type $XXXm$ or $XmXX$ is called **the second mixed problem.**

Hence, the general case is divided into two separate types of problems:

a) the first mixed problem P_{XXX} or P_{XmXX};

b) the second mixed problem P_{XXXm} or P_{XmXXm}.

The equation (1) has several conditions attached (strictly) in the following order: 1) the initial conditions; 2a) and / or 2b) the limit conditions (or the border conditions, the surface conditions etc.):

$$1)\ u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x),\ x \in D,\ D \subset R\ \text{(domain)}. \qquad (2)$$

These represents the initial form of the rope and the speed of the points on the arrow u.

$$2a)\ u(x,t) = h(x,t),\ x \in \partial D\ \text{(for the first mixed problem)} \qquad (3a)$$

$$2b)\ \frac{\partial u}{\partial x}(x,t) = h_1(x,t),\ x \in \partial D\ \text{(for the second mixed problem)} \qquad (3b)$$

where ∂D is the border of D and $h(x,t),\ h_1(x,t)$ designate a set of given functions. The relations (1), (2), (3a) or (1), (2), (3b) represent three relations in a fixed order XXX.

Remark 1. For the first mixed problem, the limit conditions **do not contain** the derivative $\partial u / \partial x$. For the second mixed problem, the limit conditions **contain** the derivative $\partial u / \partial x$.

The functions $F(x,t),\ f(x),\ g(x),\ h(x,t),\ h_1(x,t)$ could be the null functions (and we denote it by the sign O) or non-null functions (and we denote it by the sign N). So, the letter X becomes O or N and by selection, we can have a lot of combinations over the following elements:

$$a(x)\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}+\alpha(t)\frac{\partial^2 u}{\partial t^2}+\beta(t)\frac{\partial u}{\partial t}+[c(x)+\gamma(t)]u=0 \quad (4)$$

$$\text{or } u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0,\ x\in D,\ D\subset R \text{ (domain)} \quad (5)$$

$$\text{or } u(x,t)=0,\ \text{and / or } \frac{\partial u}{\partial x}(x,t)=0\ \ x\in\partial D. \quad (6)$$

Now we use a comparison with the figures 0,1, 2, 3, 4, 5, 6, 7 written in the base 2, which uses three fixed positions and three figures 0 and 1: 000, 001, 010, 011, 100, 101, 110, 111. Due to this comparison the equations (1),…,(6) generate 8 different problems, denoted by:

$$P_{OOO},\ P_{OON},\ P_{ONO},\ P_{ONN},\ P_{NOO},\ P_{NON},\ P_{NNO},\ P_{NNN}. \quad (7)$$

The problem P_{OOO} defined by (4), (5), (6) has only the null solution $u(x,t)=0$ and it will be excluded in the future. Hence we have to solve the remaining seven problems.

The most complete problem is P_{NNN} defined by (1), (2), (3a), or (1), (2), (3b). In order to solve P_{NNN} we use 4 steps. Each step solves its special problem.

Step 1. We solve the problem P_{ONO}: (4), (2), (6) and obtain the solution $u_{ONO}(x,t)$. The solving is based on **the variable separating method** (VSM), where we look for a solution of the form $u(x,t)=X(x)T(t)$.

Step 2. We solve the problem P_{NOO}: (1), (5), (6) and obtain the solution $u_{NOO}(x,t)$. The solving is based on **the Duhamel's principle**. Here we traverse several appropriate stages.

Step 2.1. We construct a new problem by modifying the initial conditions (2).

For a fixed value s of the variable t we find the solution $v(x,t,s)$ of the problem P^*_{ONO} defined by (4), (2*), (6), where (2*) has the form:

$$v(x,t,s)_{|t=0}=0,\ \frac{\partial v(x,t,s)}{\partial t}\Big|_{t=0}=F(x,s) \quad (2^*)$$

The known function $F(x,s)$ is on the place of the function $g(x)$. The solution $v(x,t,s)$ is obtained by *VSM*.

Step 2.2. Use the solution $v(x,t,s)$ and find the solution $u_{NOO}(x,t)$ of the problem P_{NOO} ([3], page 269):

$$u_{NOO}(x,t) = \int_0^t v(x,t-x,s)\,ds. \tag{8}$$

Step 3. We solve the problem P_{NNO} : (1), (2), (6) and obtain the solution $u_{NNO}(x,t)$, where:

$$u_{NNO}(x,t) = u_{ONO}(x,t) + u_{NOO}(x,t) \tag{9}$$

([3], page 265).

Step 4. We solve the problem P_{NNN} : (1), (2), (3). In order to obtain the solution $u_{NNN}(x,t)$ it is sufficiently to find a special function $w = w(x,t)$.

Step 4.1. We construct and solve a differential problem by using the limit conditions (3):

$$\frac{d^k w(x,t)}{dx^k} = 0, \quad w(x,t) = h(x,t). \tag{10}$$

The value of $k \in N$ depends on the complexity of the limit conditions (3).

Step 4.2. The solution $u_{NNN}(x,t)$ has the form:

$$u_{NNN}(x,t) = w(x,t) + u_{NNO}(x,t). \tag{11}$$

The reduced case. The equation (1) has attached only the initial conditions of type (2) or (5). Then one obtains the particular problems denoted P_{ON}, P_{NO}, P_{NN}.

In the next sections we discuss separately each problem.

2. The solutions for the reduced case P_{XX}. Application to the vibrating rope with infinite length

There are two types of the vibrating rope: with infinite length ($x \in R$) and with finite length ($0 \le x \le L$). In this section we deal with the infinite length.

The mathematical model of the vibrating rope uses the equation:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=0 \text{ or } \frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)=0,\ x\in R,\ t\geq 0,\ a>0.$$

Remark 2. The above partial differential equations are of the **hyperbolic type.**

Proposition 2.1. Model ON. The problem P_{ON} (the Cauchy problem) (12), (13):

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=0,\ a>0 \tag{12}$$

$$u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x),\ x\in R \tag{13}$$

has the solution:

$$u(x,t)=\frac{1}{2}[f(x-at)+f(x+at)]+\frac{1}{2a}\int_{x-at}^{x+at}g(\lambda)d\lambda. \tag{14}$$

The relation (14) is called the first D' Alembert – Euler formula.

Proposition 2.2. Model NN. The problem P_{NN} (the Cauchy problem) (15), (13):

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)=0,\ a>0 \tag{15}$$

has the solution:

$$u(x,t)=\frac{1}{2}[f(x-at)+f(x+at)]+\frac{1}{2a}\int_{x-at}^{x+at}g(\lambda)d\lambda+$$
$$+\frac{a}{2}\int_0^t ds\int_{x-a(t-s)}^{x+a(t-s)}F(\lambda,s)d\lambda. \tag{16}$$

The relation (16) is called the second D' Alembert-Euler formula.

Proposition 2.3. Model NO. The problem P_{NO} (the Cauchy problem) (15), (17):

$$u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0,\ x\in R \tag{17}$$

has the solution:

$$u(x,t)=\frac{a}{2}\int_0^t ds\int_{x-a(t-s)}^{x+a(t-s)}F(\lambda,s)d\lambda. \tag{16}$$

Remark 3. For the vibrating rope with infinite length we have solved the models *ON, NN, NO*.

3. The solutions for general case P_{XXX}. Application to the vibrating rope with finite length *L*

The solving of the 7 problems *XXX* mentioned above is done in a special (appropriate) order. All the proofs and all the solutions could be found (in detail) in the monograph [NP 1996].

Now we deal with the vibrating rope having the finite length *L*.

3.1. The first mixed problem

Proposition 3.1.1. Model 1 ONO. The problem P_{ONO} defined by (19), (20) and (21):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0,\ \ 0 \le x \le L,\ t \ge 0,\ a > 0 \tag{19}$$

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x),\ \ 0 \le x \le L \tag{20}$$

$$u(0,t) = 0,\ \ u(L,t) = 0 \tag{21}$$

has the solution:

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \cos\frac{na\pi}{L}t + B_n \sin\frac{na\pi}{L}t]\sin\frac{n\pi}{L}x \tag{22}$$

where:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx,\ \ B_n = \frac{2}{na\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx,\ \ n \ge 1. \tag{23}$$

Proof. see [3], page 111.

Remark 4. For a centred problem P_{ONO}, with $-\frac{L}{2} \le x \le \frac{L}{2}$, the solution $u(x,t)$ has the form:

$$u(x,t) = \sum_{n=0}^{\infty}[A_n \cos\frac{(2n+1)a\pi}{L}t + B_n \sin\frac{(2n+1)a\pi}{L}t]\sin\frac{(2n+1)\pi}{L}x$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{(2n+1)\pi}{L}x\,dx,$$

$$B_n = \frac{2}{(2n+1)a\pi}\int_{-L/2}^{L/2} g(x)\sin\frac{(2n+1)\pi}{L}x\,dx,\ \ n \ge 0.$$

Proposition 3.1.2. Model 2 NNO. The first mixed problem P_{NNO} is defined by (24), (25), (21):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + F(x,t) = 0,\ \ 0 \le x \le L,\ \ t \ge 0 \tag{24}$$

$$u(x,0) = f(x),\ \ \frac{\partial u}{\partial t}(x,0) = g(x)\ \text{ (the initial conditions)} \tag{25}$$

$$u(0,t) = 0,\ \ u(L,t) = 0\ \text{ (the limit conditions).} \tag{21}$$

The solution $u(x,t)$ of the problem P_{NNO} can be obtained by using the Step 1 and Step 2 (see above) or it is obtained directly by the formulas:

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \cos\frac{na\pi}{L}t + B_n \sin\frac{na\pi}{L}t + T_n(t)]\sin\frac{n\pi}{L}x \tag{26}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx,\ \ B_n = \frac{2}{na\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx,\ \ n \ge 1 \tag{27}$$

and $T_n(t)$ is the solution of the Cauchy problem:

$$T_n^{''}(t) + \left(\frac{na\pi}{L}\right)^2 T_n(t) = a^2 b_n(t),\ \ b_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi}{L}x\,dx \tag{28}$$

$$T_n(0) = 0,\ \ T_n^{'}(0) = 0,\ \ n \ge 1. \tag{29}$$

Proof. see [3], page 113.

Proposition 3.1.3. Model 3 NNN. The first mixed problem P_{NNN} is defined by (24), (25), (30), where (30) has the general form:

$$u(0,t) = \varphi(t),\ \ u(L,t) = \psi(t),\ \ \varphi,\psi \in C^{(2)}\ \text{ (the limit conditions).} \tag{30}$$

We reduce the model P_{NNN} by using a special model P_{NNO}.

The solution $u(x,t)$ of the problem P_{NNN} : (24), (25), (30) has the form:

$$u(x,t) = \frac{x}{L}[\psi(t) - \varphi(t)] + \varphi(t) + v(x,t) \tag{31}$$

where the function $v(x,t)$ is the solution of the special P_{NNO} problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + F(x,t) - \frac{1}{a^2}\left[\frac{x}{L}[\psi''(t) - \varphi''(t)] + \varphi''(t)\right] = 0 \tag{32}$$

$$\begin{cases} v(x,0) = f(x) - \frac{x}{L}[\psi(0) - \varphi(0)] - \varphi(0) \\ \frac{\partial v}{\partial t}(x,0) = g(x) - \frac{x}{L}[\psi'(0) - \varphi'(0)] - \varphi'(0) \end{cases} \quad (33)$$

$$v(0,t) = 0, \; v(L,t) = 0. \quad (34)$$

The solution $v(x,t)$ is obtained by the model 2, P_{NNO}, where the functions $F(x), f(x), g(x)$ have the new special forms given by (32) and (33), respectively.

Proof. see [3], page 117.

Proposition 3.1.4. Model 4. NON. The first mixed problem P_{NON} is defined by (24), (35),(30), where *f* and *g* are the null functions:

$$u(x,0) = 0, \; \frac{\partial u}{\partial t}(x,0) = 0 \text{ (the initial conditions).} \quad (35)$$

This model is a particular case of the model NNN (model 3).

The solution $u(x,t)$ of the problem P_{NON} : (24), (35), (30) has the form:

$$u(x,t) = \frac{x}{L}[\psi(t) - \varphi(t)] + \varphi(t) + v(x,t) \quad (36)$$

where the function $v(x,t)$ is the solution of the special P_{NNO} problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + F(x,t) - \frac{1}{a^2}\left[\frac{x}{L}[\psi''(t) - \varphi''(t)] + \varphi''(t)\right] = 0 \quad (37)$$

$$\begin{cases} v(x,0) = -\frac{x}{L}[\psi(0) - \varphi(0)] - \varphi(0) \\ \frac{\partial v}{\partial t}(x,0) = -\frac{x}{L}[\psi'(0) - \varphi'(0)] - \varphi'(0) \end{cases} \quad (38)$$

$$v(0,t) = 0, \; v(L,t) = 0. \quad (39)$$

The solution $v(x,t)$ is obtained by the model 2, P_{NNO}, where the functions $F(x), f(x), g(x)$ have the new special forms given by (37) and (38), respectively.

Proof. see [3], page 117.

Proposition 3.1.5 Model 5 NOO. The first mixed problem P_{NOO} is defined by (24), (35), (21). The solution $u(x,t)$ can be obtained by two methods.

Method 1. We apply the step 2 from and the Duhamel's principle presented in the section 1.

Method 2. We reduce the model NNO (with f and g as the null functions) to the model NOO.

Then $A_n = 0, B_n = 0, n \geq 1$ and we obtain:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{L} x \tag{40}$$

where $T_n(t)$ is the solution of the Cauchy problem (41), (42):

$$T_n^{"}(t) + \left(\frac{na\pi}{L}\right)^2 T_n(t) = a^2 b_n(t),\ \ b_n(t) = \frac{2}{L}\int_0^L F(x,t) \sin \frac{n\pi}{L} x\, dx \tag{41}$$

$$T_n(0) = 0,\ \ T_n^{'}(0) = 0,\ \ n \geq 1. \tag{42}$$

Proposition 3.1.6. Model 6 OON. The first mixed problem P_{OON} is defined by (19), (35), (30). We extend the model OON to the model 4 NON, by taking $F(x,t)$ as the null function in the equation (37).

Proposition 3.1.7. Model 7 ONN. The first mixed problem P_{ONN} is defined by (19), (25), (30). We extend the model ONN to the model 3 NNN, by taking $F(x,t)$ as the null function in the equation (32).

Remark 5. Up to now we have solved 7 mixed problems of the first type, in the following order: ONO, NNO, NNN, NON, NOO, OON, ONN.

3.2. The modified first mixed problem

Proposition 3.2.1. Model 1 NmNO. The modified first mixed problem P_{NmNO} is defined by (43), (20), (21), where the modified equation (43) has the form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + bu + F(x,t) = 0,\ \ 0 \leq x \leq L,\ \ b \in R,\ \ t \geq 0. \tag{43}$$

The solution $u(x,t)$ of the problem P_{NmON} : (43), (20), (21) has the form:

$$u(x,t)=\frac{2}{L}\sum_{k=1}^{\infty}\sin\frac{k\pi x}{L}\cos(\omega_k t)\int_0^L f(\xi)\sin\frac{k\pi\xi}{L}d\xi+$$

$$+\frac{2}{L}\sum_{k=1}^{\infty}\frac{1}{\omega_k}\sin\frac{k\pi x}{L}\sin(\omega_k t)\int_0^L g(\xi)\sin\frac{k\pi\xi}{L}d\xi+$$

$$+\frac{2a^2}{L}\sum_{k=1}^{\infty}\frac{1}{\omega_k}\sin\frac{k\pi x}{L}\int_0^L \sin\frac{k\pi\xi}{L}d\xi\int_0^t F(\xi,\tau)\sin[\omega_k(t-\tau)\,d\tau \qquad (44)$$

where $b>0$ and $\omega_k=a\sqrt{\frac{k^2\pi^2}{L^2}-b}$, with the condition $\frac{k^2\pi^2}{L^2}-b>0, k\geq 1$.

Proof. see [3], page 119. One uses the inverse Laplace transform $£^{-1}$ (by Mellin-Fourier formula) or one uses the Green function applied to the transformed problem.

Now we extend the above model NmNO to the non-homogenous limit conditions (30).

Proposition 3.2.2. Model 2 NmNN. The modified first mixed problem P_{NmNN} is defined by (43), (20), (30). The solution $u(x,t)$ of the problem P_{NmNN} : (43), (20), (30) has the form:

$$u(x,t)=\frac{x}{L}[\psi(t)-\varphi(t)]+\varphi(t)+v(x,t)$$

where the function $v(x,t)$ is the solution of the special problem P_{NmNO} :

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}+bv+F(x,t)-\frac{1}{a^2}\left[\frac{x}{L}[\psi''(t)-\varphi''(t)]+\varphi''(t)\right]+$$

$$+b\left[\frac{x}{L}[\psi(t)-\varphi(t)]+\varphi(t)\right]=0 \qquad (45)$$

$$\begin{cases} v(x,0)=f(x)-\frac{x}{L}[\psi(0)-\varphi(0)]-\varphi(0) \\ \frac{\partial v}{\partial t}(x,0)=g(x)-\frac{x}{L}[\psi'(0)-\varphi'(0)]-\varphi'(0) \end{cases} \qquad (46)$$

$$v(0,t)=0,\ v(L,t)=0. \qquad (47)$$

Proof. see [3], page 123. In order to obtain the solution $v(x,t)$ we apply the above proposition 3.2.1.

Remark 6. For the modified first mixed problem we have solved two models: NmNO, NmNN.

3.3. The second mixed problem

On the position number three, the first mixed problem contains only the limit conditions for function u, like $u(0,t)$ and $u(L,t)$.

In comparison with the first mixed problem, the second mixed problem has some different limit conditions X on the position number three, where appears at least one condition for the derivative $\partial u/\partial x$. So, for the vibrating rope, the limit conditions could be:

a) $u(0,t)=\varphi(t)$, $\dfrac{\partial u}{\partial x}(L,t)=\psi(t)$; we denote it by X_L (only derivative in $x=L$);

b) $\dfrac{\partial u}{\partial x}(0,t)=\varphi(t)$, $u(L,t)=\psi(t)$; we denote it by X_0 (only derivative in $x=0$);

c) $\dfrac{\partial u}{\partial x}(0,t)=\varphi(t)$, $\dfrac{\partial u}{\partial x}(L,t)=\psi(t)$; we denote it by X_{0L} (derivative in $x=0$ and $x=L$).

The functions φ and/or ψ could be null or non-null functions.

Proposition 3.3.1. Model 1 NNO$_L$. The second mixed problem P_{NNO_L} is defined by:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)=0;\ u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x) \text{ and}$$

$$u(0,t)=0,\ \frac{\partial u}{\partial x}(L,t)=0. \tag{48}$$

The solution $u(x,t)$ has the form:

$$u(x,t)=\frac{2}{L}\sum_0^\infty \sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L f(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi+$$

$$+\frac{4}{\pi a}\sum_0^\infty \frac{1}{2k+1}\sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L g(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi+$$

$$+$$

$$+\frac{4a}{\pi}\sum_0^\infty \frac{1}{2k+1}\sin\frac{\{2k+1)\pi x}{2L}\int_0^t \sin\frac{(2k+1)\pi\xi}{2L}d\xi\int_0^t F(\xi,\tau)\sin\frac{(2k+1)\pi a(t-\tau)}{2L}d\tau.$$

Proof. see [3], page 127.

Remark 7. Because the convolution product is commutative, the last integral from the above formula could be written also on the form:

$$\int_0^t F(\xi,t-\tau)\sin\frac{(2k+1)\pi a\tau}{2L}d\tau.$$

Proposition 3.3.2. Model 2 ONO$_L$. The second mixed problem P_{ONO_L} is defined by:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=0;\ u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x)\ \text{and (48).}$$

The solution $u(x,t)$ has the form:

$$u(x,t)=\frac{2}{L}\sum_0^\infty \sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L f(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi+$$

$$+\frac{4}{\pi a}\sum_0^\infty \frac{1}{2k+1}\sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L g(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi.$$

Proposition 3.3.3. Model 3 NNO$_0$. The second mixed problem $P_{NNO\,0}$ is defined by:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)=0;\ u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x)\ \text{and (49),}$$

where
$$\frac{\partial u}{\partial x}(0,t)=0,\ u(L,t)=0. \tag{49}$$

The solution $u(x,t)$ has the form:

$$u(x,t)=\frac{2}{L}\sum_0^\infty \sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L f(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi +$$

$$+\frac{4}{\pi a}\sum_0^\infty\frac{1}{2k+1}\sin\frac{\{2k+1)\pi x}{2L}\sin\frac{(2k+1)\pi at}{2L}\int_0^L g(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi+$$

$$+$$

$$+\frac{4a}{\pi}\sum_0^\infty\frac{1}{2k+1}\sin\frac{\{2k+1)\pi x}{2L}\int_0^t\sin\frac{(2k+1)\pi\xi}{2L}d\xi\int_0^t F(\xi,\tau)\sin\frac{(2k+1)\pi a(t-\tau)}{2L}d\tau.$$

Proof. see [3], page 127.

Proposition 3.3.4. Model 4 NNN$_0$. The second mixed problem $P_{NNN\,0}$ is defined by:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)=0;\ u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x)$$ and (50), where

$$\frac{\partial u}{\partial x}(0,t)=\varphi(t),\ u(L,t)=\psi(t),\ \varphi,\psi\in C^{(2)}. \qquad (50)$$

The solution $u(x,t)$ has the form $u(x,t)=(x-L)\varphi(t)+\psi(t)+v(x,t)$ where $v(x,t)$ is the solution of the following associated problem $P_{NNO\,0}$ (from model 3):

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}+F(x,t)-\frac{1}{a^2}\left[(x-L)\varphi''(t)+\psi''(t)\right]=0$$

$$v(x,0)=f(x)-(x-L)\varphi(0)-\psi(0),\ \frac{\partial v}{\partial t}(x,0)=g(x)-(x-L)\varphi'(0)-\psi'(0)$$

$$\frac{\partial v}{\partial x}(0,t)=0,\ v(L,t)=0.$$

Proof. see [3], page 134.

Proposition 3.3.5. Model 5 ONN$_0$. The second mixed problem $P_{ONN\,0}$ is defined by:

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=0;\ u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=g(x)$$ and (50).

The solution $u(x,t)$ has the form $u(x,t) = (x-L)\varphi(t) + \psi(t) + v(x,t)$ where $v(x,t)$ is the solution of the following associated problem $P_{NNO\,0}$ (from model 3):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{a^2}\left[(x-L)\varphi''(t) + \psi''(t)\right] = 0$$

$$v(x,0) = f(x) - (x-L)\varphi(0) - \psi(0),$$

$$\frac{\partial v}{\partial t}(x,0) = g(x) - (x-L)\varphi'(0) - \psi'(0)$$

$$\frac{\partial v}{\partial x}(0,t) = 0,\ \ v(L,t) = 0.$$

Proof. see [3], page 137.

Proposition 3.3.6. Model 6. NNO$_{0L}$. The second mixed problem $P_{NNO\,0L}$ is defined by:

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + F(x,t) = 0;\ u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x)$ and (51), where

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0. \tag{51}$$

The solution $u(x,t)$ has the form:

$$u(x,t) = \frac{1}{L}\int_0^L \left[f(\xi) + t g(\xi) + a^2 \int_0^t (t-\tau) F(\xi,\tau)\, d\tau \right] d\xi +$$

$$\frac{2}{L}\sum_{k=1}^{\infty} \cos\frac{k\pi x}{L} \cos\frac{k\pi a t}{L} \int_0^L f(\xi) \cos\frac{k\pi\xi}{L}\, d\xi +$$

$$+\frac{2}{a\pi}\sum_{k=1}^{\infty} \frac{1}{k} \cos\frac{k\pi x}{L} \sin\frac{k\pi a t}{L} \int_0^L g(\xi) \cos\frac{k\pi\xi}{L}\, d\xi +$$

$$+\frac{2a}{\pi}\sum_{k=1}^{\infty} \frac{1}{k} \cos\frac{k\pi x}{L} \int_0^L d\xi \int_0^t F(\xi,\tau) \cos\frac{k\pi\xi}{L} \sin\frac{k\pi a(t-\tau)}{L}\, d\tau.$$

Proof. The solution $u(x,t)$ is obtained by the using the Laplace transform £, the inverse Laplace transform £$^{-1}$ and the Green function $G(x,\xi,s)$ applied to the transformed problem [3], page 139.

Proposition 3.3.7. Model 7 NNN$_{0L}$. The second mixed problem $P_{NNN\ 0L}$ is defined by:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + F(x,t) = 0;\ \ u(x,0) = f(x),\ \ \frac{\partial u}{\partial t}(x,0) = g(x) \text{ and } (52),$$

where:

$$\frac{\partial u}{\partial x}(0,t) = \varphi(t),\ \frac{\partial u}{\partial x}(L,t) = \psi(t). \tag{52}$$

The solution $u(x,t)$ has the form $u(x,t) = (x-L)\varphi(t) + \psi(t) + v(x,t)$ where $v(x,t)$ is the solution of the following associated problem $P_{NNO\ 0L}$ (from model 6):

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + F(x,t) + \frac{\psi(t)-\varphi(t)}{L} - \frac{1}{a^2}\left[\frac{x^2}{2L}[\psi''(t) - \varphi''(t)] + x\varphi''(t)\right] = 0$$

$$\begin{cases} v(x,0) = f(x) - \dfrac{x^2}{2L}[\psi(0) - \varphi(0)] - x\varphi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - \dfrac{x^2}{2L}[\psi'(0) - \varphi'(0)] - x\varphi'(0) \end{cases}$$

$$\frac{\partial v}{\partial x}(0,t) = 0,\ \frac{\partial v}{\partial x}(L,t) = 0.$$

Proof. see [3], page 138.

Remark 8. Up to now we have solved 7 mixed problems of the second type, in the following order: NNO_L, ONO_L, NNO_0, NNN_0, ONN_0, NNO_{0L}, NNN_{0L}.

3.4. The modified second mixed problem

The modification is related with the partial differential equation of the second order i.e.:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + bu + F(x,t) = 0,\ \ 0 \le x \le L,\ \ b \in R,\ \ t \ge 0. \tag{53}$$

In order to solve the modified mixed problem we use an unitary method based on the Green function. We take into account three cases related with the point where one computes the derivative $\partial u / \partial x$: $x = L$ or $x = 0$, or both points together.

Proposition 3.4.1. Model 1 NmNO$_L$. The modified second mixed problem P_{NmNO_L} is defined by (53), (54) and (55), where:

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x) \tag{54}$$

$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0. \tag{55}$$

The solution $u(x,t)$ has the form:

$$u(x,t) = \frac{2}{L}\sum_{k=0}^{\infty} \sin\frac{(2k+1)\pi x}{2L}\cos\omega_k t\int_0^L f(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi +$$

$$+\ \frac{2}{L}\sum_{k=0}^{\infty} \sin\frac{(2k+1)\pi x}{2L}\frac{\sin_k t}{\omega_k}\int_0^L g(\xi)\sin\frac{(2k+1)\pi\xi}{2L}d\xi +$$

$$+\ \frac{2\pi^2}{L}\sum_{k=0}^{\infty}\frac{1}{\omega_k}\sin\frac{(2k+1)\pi x}{2L}\int_0^L \sin\frac{(2k+1)\pi\xi}{2L}d\xi\int_0^t F(\xi,\tau)\sin\omega_k(t-\tau)\,d\tau$$

where:

$$\omega_k = a\sqrt{\left[\frac{(2k+1)\pi}{2L}\right]^2 - b},\ \left[\frac{(2k+1)\pi}{2L}\right]^2 - b > 0,\ k \geq 0. \tag{56}$$

Proof. see [3], page 145.

Proposition 3.4.2. Model 2. NmNN$_L$. The modified second mixed problem P_{NmNN_L} is defined by (53), (54) and (57), where:

$$u(0,t) = \varphi(t),\ \frac{\partial u}{\partial x}(L,t) = \psi(t). \tag{57}$$

The solution $u(x,t)$ has the form $u(x,t) = \varphi(t) + x\psi(t) + v(x,t)$, where $v(x,t)$ is the solution of the following problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + bv + F(x,t) - \frac{1}{a^2}\left[\varphi''(t) + x\psi''(t)\right] + b\left[x\psi(t) + \varphi(t)\right] = 0.$$

$$\begin{cases} v(x,0) = f(x) - x\psi(0) - \varphi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - x\psi'(0) - \varphi'(0) \end{cases}$$

$$v(0,t) = 0,\ \frac{\partial v}{\partial x}(L,t) = 0.$$

Proof. see [3], page 144. The solution $v(x,t)$ is obtained by the model 1 $P_{NmNO\,L}$ from proposition 3.4.1.

Proposition 3.4.3. Model 3 NmNO $_0$. The modified second mixed problem $P_{NmNO\,0}$ is defined by (53), (54) and (58), where:

$$\frac{\partial u}{\partial x}(0,t) = 0,\ u(L,t) = 0. \tag{58}$$

The solution $u(x,t)$ has the form:

$$u(x,t) = \frac{2}{L}\sum_{k=0}^{\infty}\cos\frac{(2k+1)\pi x}{2L}\cos\omega_k t\int_0^L f(\xi)\cos\frac{(2k+1)\pi\xi}{2L}d\xi +$$

$$+\frac{2}{L}\sum_{k=0}^{\infty}\cos\frac{(2k+1)\pi x}{2L}\frac{\sin\omega_k t}{\omega_k}\int_0^L g(\xi)\cos\frac{(2k+1)\pi\xi}{2L}d\xi +$$

$$+\frac{2\pi^2}{L}\sum_{k=0}^{\infty}\frac{1}{\omega_k}\cos\frac{(2k+1)\pi x}{2L}\int_0^L\cos\frac{(2k+1)\pi\xi}{2L}d\xi\int_0^t F(\xi,\tau)\sin\omega_k(t-\tau)\,d\tau$$

where ω_k is defined by (56).

Proof. see [3], page 149.

Proposition 3.4.4. Model 4 NmNN $_0$. The modified second mixed problem $P_{NmNN\,0}$ is defined by (53), (54) and (59), where:

$$\frac{\partial u}{\partial x}(0,t) = \varphi(t),\ u(L,t) = \psi(t). \tag{59}$$

The solution $u(x,t)$ has the form $u(x,t) = (x-L)\varphi(t) + \psi(t) + v(x,t)$, where $v(x,t)$ is the solution of the following problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + bv + F(x,t) - \frac{1}{a^2}\left[(x-L)\varphi''(t) + \psi''(t)\right] +$$

$$+ b\left[\psi(t) + (x-L)\varphi(t)\right] = 0$$

$$\begin{cases} v(x,0) = f(x) - (x-L)\,\varphi(0) - \psi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - (x-L)\varphi'(0) - \psi'(0) \end{cases}$$

$$\frac{\partial v}{\partial x}(0,t) = 0,\ \ v(L,t) = 0.$$

Proof. see [3], page 148. The solution $v(x,t)$ is obtained by the model 3 P_{NmNO_0} from proposition 3.4.3.

Proposition 3.4.5. Model 5 NmNO$_{0L}$. The modified second mixed problem $P_{NmNO_{0L}}$ is defined by (53), (54) and (60), where:

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \ \frac{\partial u}{\partial x}(L,t) = 0. \tag{60}$$

The solution $u(x,t)$ has the form:

$$u(x,t) = \frac{ch\left(a\sqrt{b}\,t\right)}{L}\int_0^L f(\xi)\,d\xi + \frac{sh\left(a\sqrt{b}\,t\right)}{L\,a\sqrt{b}}\int_0^L g(\xi)\,d\xi +$$

$$+\frac{2}{L}\sum_{k=1}^{\infty}\cos\frac{k\pi x}{L}\cos\left(\omega_k t\right)\int_0^L f(\xi)\cos\frac{k\pi x}{L}d\xi +$$

$$+\frac{2}{L}\sum_{k=1}^{\infty}\cos\frac{k\pi x}{L}\frac{\sin\left(\omega_k t\right)}{\omega_k}\int_0^L g(\xi)\cos\frac{k\pi x}{L}d\xi +$$

$$+a^2\int_0^L d\xi\int_0^t F(\xi,t-\tau)\left[\frac{sh\left(a\sqrt{b}\,\tau\right)}{L\,a\sqrt{b}} + \frac{2}{L}\sum_{k=1}^{\infty}\cos\frac{k\pi x}{L}\cos\frac{k\pi\xi}{L}\frac{\sin\left(\omega_k\tau\right)}{\omega_k}\right]d\tau$$

where $\omega_k = a\sqrt{\left(\frac{k\pi}{L}\right)^2 - b},\ b > 0,\ \frac{k^2\pi^2}{L^2} - b > 0,\ k \ge 1.$

Proof. see [3], page 151.

Proposition 3.4.6. Model 6 NmNN$_{0L}$. The modified second mixed problem $P_{NmNN_{0L}}$ is defined by (53), (54) and (61), where:

$$\frac{\partial u}{\partial x}(0,t) = \varphi(t),\ \ \frac{\partial u}{\partial x}(L,t) = \psi(t). \tag{61}$$

The solution $u(x,t)$ has the form $u(x,t) = \frac{x^2}{2L}[\psi(t) - \varphi(t)] +$ $+ x\varphi(t) + v(x,t),$ where $v(x,t)$ is the solution of the following problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + bv + F(x,t) + \frac{\psi(t) - \varphi(t)}{L} -$$

$$-\frac{1}{a^2}\left[\frac{x^2}{2L}[\psi''(t) - \varphi''(t)] + x\varphi''(t)\right] + b\left[\frac{x^2}{2L}[\psi(t) - \varphi(t)] + x\varphi(t)\right] = 0$$

$$\begin{cases} v(x,0) = f(x) - \dfrac{x^2}{2L}[\psi(0) - \varphi(0)] - x\varphi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - \dfrac{x^2}{2L}[\psi'(0) - \varphi'(0)] - x\varphi'(0) \end{cases}$$

$$\frac{\partial v}{\partial x}(0,t) = 0, \ \frac{\partial v}{\partial x}(L,t) = 0.$$

Proof. see [3], page 150. The solution $v(x,t)$ is obtained by the model 5 $P_{NmNO_{0\,L}}$ from proposition 3.4.5.

Remark 9. We have solved 6 modified mixed problems of the second type, in the following order: $NmNO_L$, $NmNN_L$, $NmNO_0$, $NmNN_0$, $NmNO_{0L}$, $NmNN_{0L}$.

4. The equation of the telegraphers

Now we return back to the equation (1) and we consider it has the form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + bu + 2c\frac{\partial u}{\partial x} + 2d\frac{\partial u}{\partial t} + F(x,t) = 0, \ 0 \le x \le L, t \ge 0. \quad (62)$$

The equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + bu + 2d\frac{\partial u}{\partial t} + F(x,t) = 0 \quad (63)$$

is called the equation of the telegraphers.

Again we will study several problems. We denote these problems by T1XXX (**the first mixed problem of telegraphers**) and T2XX (**the second mixed problem of telegraphers**), where *X* could be *N* or *O*.

4.1. The first mixed problem of telegraphers

Proposition 4.1.1 Model 1 T1NNO. The first mixed problem of telegraphers P_{T1NNO} is defined by equation (63) and the conditions (64) and (65), where:

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x) \tag{64}$$

$$u(0,t) = 0,\ u(L,t) = 0. \tag{65}$$

The solution $u(x,t)$ of P_{T1NNO} has the form $u(x,t) = e^{a^2 dt} v(x,t)$, where $v(x,t)$ is the solution of the problem P_{NmNO} having the form:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + (b + a^2 d^2) v + e^{-a^2 dt} F(x,t) = 0$$

(attention: here dt is a real product and not a differential dt)

$$v(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x) - a^2 d\ f(x);\ v(0,t) = 0,\ v(L,t) = 0.$$

By solving the problem P_{NmNO} we obtain:

$$v(x,t) = \frac{2}{L}\sum_{k=1}^{\infty} \sin\frac{k\pi x}{L}\cos(\omega_k t)\int_0^L f(\xi)\sin\frac{k\pi\xi}{L}d\xi +$$

$$+\frac{2}{L}\sum_{k=1}^{\infty} \sin\frac{k\pi x}{L}\frac{\sin(\omega_k t)}{\omega_k}\int_0^L [g(\xi) - a^2 dt]\sin\frac{k\pi\xi}{L}d\xi +$$

$$+\frac{2a^2}{L}\sum_{k=1}^{\infty}\frac{1}{\omega_k}\sin\frac{k\pi x}{L}\int_0^L \sin\frac{k\pi\xi}{L}d\xi\int_0^t e^{-a^2 d\tau}F(\xi,\tau)\sin\omega_k(t-\tau)d\tau$$

where $b + a^2 d^2 > 0$, $\omega_k = a\sqrt{\frac{k^2\pi^2}{L^2} - (b + a^2 d^2)}$, $\frac{k^2\pi^2}{L^2} - (b + a^2 d^2) > 0$, $k \geq 1$.

Proof. see [3], page 157.

Proposition 4.1.2. Model 2 T1NNN. The first mixed problem of telegraphers P_{T1NNN} is defined by (63), (64), (66), where:

$$u(0,t) = \varphi(t),\ u(L,t) = \psi(t). \quad (66)$$

The solution $u(x,t)$ of the problem P_{T1NNN} has the form $u(x,t) = \frac{x}{L}[\psi(t) - \varphi(t)] + \varphi(t) + v(x.t)$, where $v(x,t)$ is the solution of the following problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + bv + 2d\frac{\partial v}{\partial t} + F(x,t) - \frac{1}{a^2}\left[\frac{x}{L}[\psi''(t) - \varphi''(t)] + \varphi''(t)\right] +$$

$$+ b\left[\frac{x}{L}[\psi(t) - \varphi(t)] + \varphi(t)\right] + 2d\left[\frac{x}{L}[\psi'(t) - \varphi'(t)] + \varphi'(t)\right] = 0$$

$$\begin{cases} v(x,0) = f(x) - \frac{x}{L}[\psi(0) - \varphi(0)] - \varphi(0) \\ \frac{\partial v}{\partial t}(x,0) = g(x) - \frac{x}{L}[\psi'(0) - \varphi'(0)] - \varphi'(0) \end{cases}$$

$$v(0,t) = 0,\ \frac{\partial v}{\partial x}(L,t) = 0.$$

Proof. see [3], page 156. The solution $v(x,t)$ is obtained by the model 1 P_{T1NNO} from the proposition 4.1.1.

Remark 10. We have solved 2 mixed problem of telegraphers, of the first type, in the following order T1NNO and T1NNN.

4.2. The second mixed problem of telegraphers

The second mixed problem of telegraphers is denoted T2XXX. It has three different forms related with the point where the derivative $\frac{\partial u}{\partial x}$ is calculated. So, there exist three cases:

$u(0,t) = \varphi(t), \frac{\partial u}{\partial x}(L,t) = \psi(t);$ it generates the model $\mathrm{T2XXX_L}$;

$\frac{\partial u}{\partial x}(0,t)=\varphi(t), u(L,t)=\psi(t);$ it generates the model $\mathrm{T2XXX}_0$;

$\frac{\partial u}{\partial x}(0,t)=\varphi(t), \frac{\partial u}{\partial x}(L,t)=\psi(t);$ it generates the model $\mathrm{T2XXX}_{0L}$.

Proposition 4.2.1. Model 1 T2NNO $_L$. The second mixed problem of telegraphers P_{T2NNO_L} is defined by equation (63) and the conditions (64) and (67), where:

$$u(0,t)=0, \frac{\partial u}{\partial x}(L,t)=0. \qquad (67)$$

The solution $u(x,t)$ has the form $u(x,t)=e^{a^2dt}v(x,t)$, where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}+(b+a^2d^2)v+e^{-a^2dt}F(x,t)=0$$

$$v(x,0)=f(x);\ \frac{\partial v}{\partial t}(x,0)=g(x)-a^2d\,f(x);\ v(0,t)=0,\ \frac{\partial v}{\partial x}(L,t)=0.$$

Proof. see [3], page 160. The problem with the unknown function $v(x,t)$ has the form P_{NmNO_L} from the proposition 3.4.1, model 1. By using the substitution $u(x,t)=e^{a^2dt}v(x,t)$, the term $\partial v/\partial t$ vanishes from the equation (63).

Proposition 4.2.2. Model 2 T2NNN $_L$. The second mixed problem of telegraphers P_{T2NNN_L} is defined by equation (63) and the conditions (64) and (68), where:

$$u(0,t)=\varphi(t), \frac{\partial u}{\partial x}(L,t)=\psi(t). \qquad (68)$$

The solution $u(x,t)$ of the problem P_{T2NNN_L} has the form $u(x,t)=x\psi(t)+\varphi(t)+v(x,t)$, where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}+bv+2d\frac{\partial v}{\partial t}+F(x,t)-\frac{1}{a^2}[x\psi''(t)+\varphi''(t)]+$$

$$+b[x\psi(t)+\varphi(t)]+\ 2d\,[x\psi'(t)+\varphi'(t)]=0$$

$$\begin{cases} v(x,0) = f(x) - x\psi(0) - \varphi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - x\psi'(0) - \varphi'(0) \end{cases}; \; v(0,t) = 0, \; \frac{\partial v}{\partial x}(L,t) = 0.$$

Proof. see [3], page 159. The problem with the unknown function $v(x,t)$ has the form P_{T2NNO_L} from the proposition 4.2.1 model 1.

Proposition 4.2.3. Model 3 T2NNO$_0$. The second mixed problem of telegraphers P_{T2NNO_0} is defined by equation (63) and the conditions (64) and (69), where:

$$\frac{\partial u}{\partial x}(0,t) = 0, \; u(L,t) = 0. \tag{69}$$

The solution $u(x,t)$ has the form $u(x,t) = e^{a^2 dt} v(x,t)$, where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + (b + a^2 d^2)v + e^{-a^2 dt} F(x,t) = 0$$

$$v(x,0) = f(x); \; \frac{\partial v}{\partial t}(x,0) = g(x) - a^2 d\, f(x); \; \frac{\partial v}{\partial x}(0,t) = 0, \; v(L,t) = 0.$$

Proof. see [3], page 162. The problem with the unknown function $v(x,t)$ has the form P_{NmNO_0} from the proposition 3.4.3, model 3. By using the substitution $u(x,t) = e^{a^2 dt} v(x,t)$, the term $\partial v / \partial t$ vanishes from the equation (63).

Proposition 4.2.4. Model 4 T2NNN$_0$. The second mixed problem of telegraphers P_{T2NNN_0} is defined by equation (63) and the conditions (64) and (70), where:

$$\frac{\partial u}{\partial x}(0,t) = \varphi(t), \; u(L,t) = \psi(t). \tag{70}$$

The solution $u(x,t)$ of the problem P_{T2NNN_0} has the form $u(x,t) = (x-L)\varphi(t) + \psi(t) + v(x,t)$, where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + bv + 2d\frac{\partial v}{\partial t} + F(x,t) - \frac{1}{a^2}[(x-L)\varphi''(t) + \psi''(t)] +$$
$$+ b[(x-L)\varphi(t) + \psi(t)] + 2d[(x-L)\varphi'(t) + \psi'(t)] = 0$$

$$\begin{cases} v(x,0) = f(x) - (x-L)\varphi(0) - \psi(0) \\ \dfrac{\partial v}{\partial t}(x,0) = g(x) - (x-L)\varphi'(0) - \psi'(0) \end{cases}; \quad \frac{\partial v}{\partial x}(0,t) = 0, \; v(L,t) = 0.$$

Proof. see [3], page 161. The solution $v(x,t)$ can be obtained by the proposition 4.2.3, model 3.

Proposition 4.2.5. Model 5. T2NNO$_{0L}$. The second mixed problem of telegraphers $P_{T2NNO_{0L}}$ is defined by equation (63) and the conditions (64) and (71), where:

$$\frac{\partial u}{\partial x}(0,t) = 0, \; \frac{\partial u}{\partial x}(L,t) = 0. \tag{71}$$

The solution $u(x,t)$ has the form $u(x,t) = e^{a^2 dt} v(x,t)$, where the function $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + (b + a^2 d^2) v + e^{-a^2 dt} F(x,t) = 0$$

$$v(x,0) = f(x); \; \frac{\partial v}{\partial t}(x,0) = g(x) - a^2 d\, f(x); \; \frac{\partial v}{\partial x}(0,t) = 0, \; \frac{\partial v}{\partial x}(L,t) = 0.$$

NmNO0L

Proof. see [3], page 165. The solution $v(x,t)$ can be obtained by the proposition 3.4.5, model 5, for the problem $P_{NmNO_{0L}}$.

Proposition 4.2.6. Model 6 T2NNN$_{0L}$. The second mixed problem of telegraphers $P_{T2NNN_{0L}}$ is defined by equation (63) and the conditions (64) and (72), where:

$$\frac{\partial u}{\partial x}(0,t) = \varphi(t), \; \frac{\partial u}{\partial x}(L,t) = \psi(t). \tag{72}$$

The solution $u(x,t)$ of the problem $P_{T2NNN_{0L}}$ has the form $u(x,t) = \dfrac{x^2}{2L}[\psi(t) - \varphi(t)] + x\varphi(t) + v(x,t)$, where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}+bv+2d\frac{\partial v}{\partial t}+F(x,t)+\frac{\psi(t)-\varphi(t)}{L}-$$

$$-\frac{1}{a^2}\left[\frac{x^2}{2L}[\psi''(t)-\varphi''(t)]+x\varphi''(t)\right]+$$

$$+\ b\left[\frac{x^2}{2L}[\psi(t)-\varphi(t)]+x\varphi(t)\right]+\ 2d\left[\frac{x^2}{2L}[\psi'(t)-\varphi'(t)]+x\varphi'(t)\right]=0$$

$$\begin{cases} v(x,0)=f(x)-\dfrac{x^2}{2L}[\psi(0)-\varphi(0)]-x\varphi(0) \\ \dfrac{\partial v}{\partial t}(x,0)=g(x)-\dfrac{x^2}{2L}[\psi'(0)-\varphi'(0)]-x\varphi'(0) \end{cases};\ \frac{\partial v}{\partial x}(0,t)=0,\ \frac{\partial v}{\partial x}(L,t)=0.$$

Proof. see [3], page 164. The solution $v(x,t)$ can be obtained by the proposition 4.2.5, model 5, problem $P_{T2NNO_{0L}}$.

Remark 11. We have solved 6 mixed problem of telegraphers, of the second type, in the following order P_{T2NNO_L}, P_{T2NNN_L}, P_{T2NNO_0}, P_{T2NNN_0}, $P_{T2NNO_{0L}}$, $P_{T2NNN_{0L}}$.

5. A complete solved problem for a numerical example

Problem. Find the solution $u(x,t)$ for the following problem:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+u+3\frac{\partial u}{\partial t}+\cos\frac{3x}{2}-x(4+t)=0,\ \ 0\le x\le\pi, t\ge 0$$

$$u(x,0)=x, \frac{\partial u}{\partial t}(x,0)=x;\ \frac{\partial u}{\partial x}(0,t)=t+1,\ u(\pi,t)=\pi(t+1).$$

Solution. We recognize the mathematical model of the second mixed problem of telegraphers, denoted P_{T2NNN_0}. Now we describe all the steps for solving this problem ([2], page 443; [3], page 237).

a) coefficients and functions:

$$L=\pi, a=1, b=1, d=3/2;\ F(x,t)=\cos\frac{3x}{2}-x(4+t)$$

$$f(x)=x, g(x)=x, \varphi(t)=t+1, \psi(t)=\pi(t+1).$$

b) look for $u(x,t)$ having the form

$$u(x,t)=(x-L)\varphi(t)+v(x,t) \text{ i.e. } u(x,t)=x(t+1)+v(x,t)$$

(in order to oblation null limit conditions for a new problem), where $v(x,t)$ is the solution of the problem:

$$\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial t^2}+v+3\frac{\partial u}{\partial t}+\cos\frac{3x}{2}=0;\ v(x,0)=0,\frac{\partial v}{\partial t}(x,0)=0;$$

$$\frac{\partial v}{\partial x}(0,t)=0,\ v(\pi,t)=0.$$

$$L=\pi, a=1, b=1, d=3/2;\ F(x,t)=\cos\frac{3x}{2};$$

$$f(x)=0, g(x)=0, \varphi(t)=0, \psi(t)=0.$$

For the unknown function $v(x,t)$ we recognize the problem P_{T2NNO_0}.

c) A new change of function assures the elimination of the term $\partial v/\partial t$. Successively we have:

$$v(x,t)=e^{a^2dt}w(x,t) \text{ i.e. } v(x,t)=e^{3t/2}w(x,t)$$

$$\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial t^2}+\frac{13w}{4}+e^{-3t/2}\cos\frac{3x}{2}=0;\ w(x,0)=0,\frac{\partial w}{\partial t}(x,0)=0;$$

$$\frac{\partial w}{\partial x}(0,t)=0,\ w(\pi,t)=0.$$

For the unknown function $w(x,t)$ we recognize the modified second mixed problem P_{NmOO_0}, proposition 3.4.3 model e, having the data:

$$L=\pi, a=1, b=13/4;\ F(x,t)=e^{-3t/2}\cos\frac{3x}{2};$$

$$f(x)=0, g(x)=0, \varphi(t)=0, \psi(t)=0.$$

d) We solve the problem with unknown function $w(x,t)$ and we obtain successively:

$$w(x,t)=$$

$$=\frac{2\pi^2}{L}\sum_{k=0}^{\infty}\frac{1}{\omega_k}\cos\frac{(2k+1)\pi x}{2L}\int_0^L\cos\frac{(2k+1)\pi\xi}{2L}d\xi\int_0^t F(\xi,\tau)\sin\omega_k(t-\tau)\,d\tau$$

where: $\omega_k = a\sqrt{\left[\frac{(2k+1)\pi}{2L}\right]^2 - b}$, $\left[\frac{(2k+1)\pi}{2L}\right]^2 - b > 0,\ k \ge 0$ i.e.

$$w(x,t) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{\omega_k}\cos\frac{(2k+1)x}{2}\int_0^{\pi}\cos\frac{(2k+1)\xi}{2}d\xi\int_0^t\cos\frac{3\xi}{2}\sin\omega_k(t-\tau)\,d\tau$$

$$\omega_k = \frac{1}{2}\sqrt{(2k+1)^2 - 13},\ k \ge 0;\ \omega_0 = i\sqrt{3, \omega_1 = i};\ \omega_1 = i;\ \omega_k \in R, k \ge 2$$

$$\int_0^{\pi}\cos\frac{(2k+1)\xi}{2}\cos\frac{3\xi}{2}d\xi = \begin{cases}\pi/2, & k=1\\ 0, & k \ne 1\end{cases}$$

$$w(x,t) = \cos\frac{3x}{2}\int_0^t e^{-3t/2}\frac{\sin\omega_1(t-\tau)}{\omega_1}d\tau;\ \omega_1 = i;$$

$$\sin\omega_1(t-\tau) = \sin i(t-\tau) = i\,sh(t-\tau)$$

$$w(x,t) = \cos\frac{3x}{2}\int_0^t e^{-3t/2}sh(t-\tau)\,d\tau;\ \int_0^t e^{-3t/2}sh(t-\tau)\,d\tau = \int_0^t e^{-3t/2}\frac{e^{t-\tau}-e^{\tau-t}}{2}d\tau$$

$$w(x,t) = \left(\frac{1}{5}e^t + \frac{4}{5}e^{-3t/2} - e^{-t}\right)\cos\frac{3x}{2}.$$

e) Find $v(x,t)$ and then $u(x,t)$:

$$v(x,t) = \left(\frac{4}{5} + \frac{1}{5}e^{5t/2} - e^{t/2}\right)\cos\frac{3x}{2};$$

$$u(x,t) = x(t+1) + \left(\frac{4}{5} + \frac{1}{5}e^{5t/2} - e^{t/2}\right)\cos\frac{3x}{2}.\ \text{END.}$$

6. Applications to the heating propagation in a bar with infinite length or finite length *L*

The heating propagation in a bar is controlled by a partial differential equation of the form:

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0$ (homogenous case), $\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} + F(x,t) = 0$ (non-homogenous case) associated with specific conditions. These equations have the **parabolic type.**

By using our methodology illustrated in the above section of this work we are able to do a complete study in the case of parabolic type.

7. Conclusions

This big number of 33 models is a natural extension of 3 fixed elements in the following order: equation, initial conditions and limit conditions. The multitude of initial conditions and limit conditions give a big number of models.

Generally, the solution of one model is based on the solution of the precedent model. All solutions have an analytical form, based on the power series. In applications, the form of the problem conditions gives the possibility to express the power series by elementary functions.

One way to avoid the complicated analytical form is to use the numerical derivation formulas and a computer program. . But, this way is the aim of another work.

REFERENCES

[1] Vasile Brînzănescu and team, *Problems for mathematics student contests*, Polytechnic Institute, Bucharest, 1985.

[2] Gheorghe Mocică, *Problems with certain partial differential equations, part 2,* Polytechnic Institute, Bucharest, 1993.

[3] Nicolae Popoviciu, *Partial differential equations of second order. Theory and applications for equations of mathematical physics.* Military Technical Academy of Bucharest, 1996.

[4] Valter Rudner, NICOLESCU Cornelia, *Special mathematical problems*, Didactic and Pedagogic Publishing, Bucharest, 1982.

[5] Gheorghe I. Şabac, *Special mathematics*, Vol. **1** (1964), Vol. **2** (1965), Didactic and Pedagogic Publishing Bucharest.

[6] Gheorghe I. Şabac, *Special mathematics*, Vol. **1** (1981), Vol. **2** (1984), Didactic and Pedagogic Publishing Bucharest.