INVESTIGATION OF THE COMPLETE GROUP OF PERMUTATIONS S_4 BY THE CONSTRUCTION OF THE MULTI-VALUED FUNCTION THAT DESCRIBES THE COMPLEX VECTOR FIELD

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Abstract. *We investigate analytically the complete group of permutations S_4 that is responsible for the construction of the multi-valued function which describes the appropriate complex vector field. The latter is determined as over a torus in the commutative case of the group S_2, as over the ordinary complex plane in the case of the non commutative subset of the group S_4.*

The last situation is generated by the consideration of S_2 on a torus.

The aforesaid statements directly connect with the solution of Landau-Lifshitz equation when its spectral parameter varies on a torus, and complete anisotropy is taken into account.

Thus, the main goal of the present paper consists of the explicit mathematical study of the non commutative part of S_4 on the complex plane, that is raised by S_2 on a torus, and which concerns an analytical solution of the general Landau-Lifshitz equation in the framework of the more or less unified, but not universal method. The solution technique bases on the respective vector boundary Riemann problem with so called permutation matrix coefficient.

Keywords and phrases: *multi-valued function, complex vector field, non commutative case, four-dimensional complete group of permutations, vector boundary Riemann problem, diagonalization procedure.*

1. Introduction

Before we come to the reasonable explanation of the given problem explicit solution, some necessary notions must be proposed.

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At first, we associate to the permutation $\omega_m = \begin{pmatrix} 1 & \dots & m \\ j_1 & \dots & j_m \end{pmatrix}$ the $m \times m$ matrix of zeroes and ones whose k th row contains a one in the j_k th column and zeroes in other columns $(k, j_k = \overline{1, m})$. Such matrices are called permutation matrices and are in a natural one-to-one correspondence with the permutations ω_m. When instead of units the permutation matrix contains Holder functions, we come to the so called permutation matrix-function, e.g.

$$\omega_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} == (123) \leftrightarrow \Omega_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \leftrightarrow \Omega_3^* = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix},$$

and a, b, c are the Holder functions over the relevant compact / algebraic Riemann surface.

Now, let an arbitrary algebraic/compact Riemann surface R be given, and its finite genus is $r \geq 0$. We are looking for the function $f(z,u), (z,u) \in R$ (when $r = 0$, then instead of (z,u) only z is considered). This function is analytic everywhere on R, except the finite set of ramification points

$$(\alpha_i, u(\alpha_i)) \in R \ \ (i = \overline{1, n}), \tag{1.1}$$

and its functional values undergo the following m-dimensional permutations when come around the given points (1.1):

$$T_i = \begin{pmatrix} 1 & \dots & m \\ j_1^{(i)} & \dots & j_m^{(i)} \end{pmatrix} = (j_1^{(i)} \quad \dots \quad j_m^{(i)}) \quad (j_k, k = \overline{1, m}; i = \overline{1, n}). \tag{1.2}$$

Additionally, $f(z,u)$ is assumed to have a finite order at infinity.

The search of such multi-valued function $f(z,u)$, or $f(z)$ in the case of the complex plane, by its monodromy group (1.1), (1.2) [1], leads to the solution of the corresponding homogeneous vector boundary Riemann problem [2]-[4] whose boundary condition looks like

$$F^+(t,v) = M(t,v)F^-(t,v), \quad (t,v) \in L, \ D^{-1} \,|\, (F). \tag{1.3}$$

Here: $F(z,u) = \{F_j(z,u)\}_{j=1}^m$ are the unknown m-dimensional vector-valued functions that are analytic everywhere on the surface R outside the finite system of open Jordan contours L without self-intersections, and whose endpoints are the ramification points of $F(z,u)$:

$$L=\bigcup_{i=1}^{n}L_i,\ L_i=(\alpha_i,v(\alpha_i);\beta_i,v(\beta_i)),\quad L_l\bigcap L_k=\varnothing,\ (l\neq k;\ l,k,i=\overline{1,n}).\ (1.4)$$

Functions $F(z,u)$ are bounded at the endpoints of L from (1.4), extend to L H – continuously from the left and right, have a finite order at infinity, and $F^{\pm}(t,v)$ are the limited values of $F(z,u)$ at L from the left and right respectively.

Such functional class is designated as $h_0(L;R)$.

The matrix coefficient from (1.3) looks like

$$M(t,v)=\sum_{i=1}^{n}M_i\delta(t,v;L_i),\quad \delta(t,v;L_i)=\begin{cases}1,\ (t,v)\in L_i\\ 0,\ (t,v)\notin L_i\end{cases},\qquad (1.5)$$

and M_i $(i=\overline{1,n})$ is the $m\times m$ permutation matrix that is raised by the appropriate permutation (1.2).

The m-dimensional vector-divisor of infinities is D in (1.3), and $F(z,u)$ is divisible by it. When the genus of the surface R equals zero ($r=0$), then the divisibility condition $D^{-1}\,|\,(F)$ in (1.3) is missing.

Each the sought for scalar component $F_j(z,u)$ $(j=\overline{1,m})$ is the j th branch of the initially wanted multi-valued function $f(z,u)$ when $r>0$, and $f(z)$ – in the case of $r=0$.

It is well known [2] that the explicit solution of the problem (1.1)-(1.5) generates, in its turn, the construction of the relevant algebraic equation of the covering with respect to the surface R. The last fact and the research of the original vector problem (1.1)-(1.5) have a lot of industrial applications in optics, acoustics, various wave propagation and modern technical electrodynamics, as well [5]-[13].

Moreover, the most interesting case is non commutative. It means that there is no any linear transformation that is constant everywhere on the contour (1.4) and which takes all the matrices (1.5) to the diagonal form [14]. Otherwise, the vector boundary Riemann problem (1.1)-(1.5) is reduced to the corresponding system of m scalar boundary homogeneous Riemann problems on the given algebraic/compact surface R. The solutions of such scalar problems are well known [6]-[8], [12], [13] and are not of great interest. That is the main reason why in this paper, we deal only with the non commutative matrix coefficients.

Additionally, it should be noted that the proposed explicit solution of the aforesaid class of the boundary vector problems is done here in terms of the so called canonical and normal solution matrices (c.s.m. and n.s.m. correspondingly). This fact is very important, since we avoid here the habitual rather difficult investigation approach. This procedure deals with the analytical study of the respective integral equations' system that is raised by the original vector boundary Riemann problem [8].

Justifying the proposed here method of c.s.m. and n.s.m. construction for (1.1)-(1.5) problem solution, we should like to remind of the following fact. The c.s.m. and n.s.m. form the algebraic equations of the covering surfaces over the original algebraic/compact Riemann surface R. This object is principal in the inverse scattering problem and soliton theory [5], [9], [11]-[13], as well.

So, $m \times m$ matrix $\mathrm{X}(z,u), (z,u) \in R$, is the c.s.m. (canonical solution matrix) of the vector boundary problem (1.1)-(1.5) if: a) $\mathrm{X}(z,u)$ satisfies the given boundary condition; b) $\det \mathrm{X}(z,u)$ has no poles anywhere in the finite part of R and can be equal to zero only at the endpoints of the open contour L; c) the orders r_j $(j = \overline{1,m})$ of columns at infinity in the matrix $\mathrm{X}(z,u)$ can not be lessened, i.e. $\underset{\infty}{\mathrm{ord}} \det \mathrm{X}(z,u) = \sum_{j=1}^{m} r_j$. Matrix $\mathrm{X}(z,u), (z,u) \in R,$ is the n.s.m. (normal solution matrix) of the same type of vector boundary Riemann problem (1.1)-(1.5) if $\mathrm{X}(z,u)$ satisfies conditions a), b) and does not satisfy the last condition c), i.e. $\underset{\infty}{\mathrm{ord}} \det \mathrm{X}(z,u) < \sum_{j=1}^{m} r_j$.

Now, when the last necessary notions are introduced, we can move to the actual problem statement.

2. The problem statement and preliminary results

Let the complete group of permutations S_4 be given. It is well known [15] that S_4 is non commutative. In our situation, it means that not all its representatives commute in pairs.

Therefore, if S_4 is the base of the monodromy group (1.1), (1.2) [1] of the sought for multi-valued function, then we encounter one of the most difficult case of such function construction. Even the complex plane, as an initial surface of the zero genus cannot propose the explicit solution of the aforesaid problem.

Really, in terms of the so called permutation matrices [3], the last fact implies the nonexistence of the constant linear transformation that reduces all these matrices to the diagonal form [14]. In its turn, the latter shows the impossibility of the classical diagonalization procedure [4] of the respective homogeneous vector boundary Riemann problem [2] that is responsible for the construction of the appropriate multi-valued function [1]. The coefficient of the present homogeneous vector boundary Riemann problem is the permutation matrix that is in one-to-one correspondence with the group S_4.

Diagonalization means the reduction of the original vector problem to the equivalent system of scalar ones whose explicit solution is either well known, or can be done effectively by the respective technique [4]-[8].

Further, the requirements of the current industrial scientific development in majority, deal nowadays with the complete anisotropy of the studied medium that generates the urgent investigation of so popular and important nanostructures.

Such phenomena and the relevant vector fields can be described analytically only by means of the multi-valued functions over the algebraic / compact Riemann surfaces. The most interesting and non explored situation is non commutative monodromy group (1.1), (1.2) [1], as it was mentioned above.

So, **the main purpose of the given paper** *is the analytic investigation of the complete four-dimensional non commutative group of permutations* S_4*, as the base of the multi-valued function construction over the complex plane, at least. The application of these results concerns the explicit study of the Landau-Lifshitz equation whose spectral parameter varies on a torus and the complete anisotropy is taken into account [5], [11]-[13].*

This equation is principal in soliton theory, describes the dynamics of magnetization of the ferromagnetic patterns in the magnetic field and the process of the low-frequent wave propagation in ferromagnetic. These specified materials are used in the creation of the nonreciprocal wave devices [10]. Nowadays, an explicit solution of the generalized Landau-Lifshitz equation (L.-L.e.) becomes rather urgent owing to its application to the spin moment transference in the magnetic nanostructures, construction of the Magnetic Resistive Random Access Memory (MRAM) elements and magnetic logical elements as well.

Investigation of the same equation in the case of complete anisotropy [5] was done in terms of the two-sheeted covering of a torus using geometrical and physical characters of the original applied problem statement. As it was marked in [5], in other more complicated covering versions the proposed solving procedure was not effective at all. In its turn, the aforesaid mostly important current case of the L.-L.e. was studied in [11] only approximately. In papers [12], [13], the suggested general analytic approach [3], [4] deals, in particular with an explicit algebraic equation construction of the two-sheeted covering of a torus including the solution of L.-L.e.

However, the problem of an explicit study of other surfaces that can be generated by the mentioned torus' two-sheeted covering in the case of L.-L.e. was not either raised or affected at all. The last fact explains the reason why the main purpose of the suggested here research is formulated just as it is done above.

At last, from our viewpoint, the virtue of the proposed, probably deliberately simplified approach of the given article is its future leading to the effective study of the relevant problems on the surfaces of the nonzero genera.

Returning again to the studied group S_4 and all its permutations whose number equals $4! = 24$, we select and form 5 subsets of S_4 with respect to their cyclic structure. Namely,

1) 6 elements with 1 cycle of the length 4 in all 3 "similar" pairs:

$$\omega_1 = (1234), \omega_2 = (1243);$$

$$\omega_3 = (1324), \omega_4 = (1342);$$

$$\omega_5 = (1423), \omega_6 = (1432);$$

2) 1 identical element with 4 cycles of the length 1:

$$\omega_7 = (1)(2)(3)(4) = I;$$

3) 3 elements with 2 cycles of the length 2:

$$\omega_8 = (12)(34), \omega_9 = (13)(24), \ \omega_{10} = (14)(23);$$

4) 6 elements with 2 cycles of the length 1 and 1 cycle of the length 2 in all 3 "similar" pairs:

$$\omega_{11} = (1)(2)(34), \omega_{12} = (3)(4)(12);$$

$$\omega_{13} = (1)(4)(23), \omega_{14} = (2)(3)(14);$$

$$\omega_{15} = (2)(4)(13), \omega_{16} = (1)(3)(24);$$

5) 8 elements with 2 cycles of the lengths 1 and 3 correspondingly in all 4 “similar” pairs:

$$\omega_{17} = (1)(234),\ \omega_{18} = (1)(243);$$

$$\omega_{19} = (2)(134),\ \omega_{20} = (2)(143);$$

$$\omega_{21} = (3)(124),\ \omega_{22} = (3)(142);$$

$$\omega_{23} = (4)(123),\ \omega_{24} = (4)(132). \qquad (2.1)$$

Checking (2.1), it is easy to find that not all representatives inside even of one and the same subset commute in pairs.

Thus, all non trivial subsets 1), 3)-5) from (2.1) are non commutative. Only some pairs of permutations from 1), 3)-5) commute. They are the following:

1) ω_2, ω_4; ω_1, ω_6; ω_3, ω_5. The result of their multiplication is I from the subset 2). All other remained non commutative pairs, after multiplication, transform to the element from the subset 5). Thus, only 3 pairs from $C_6^2 = 15$ commute.

3) It is completely commutative subset, and each multiplying result represents the third permutation that was not included into the product as the factor. Therefore, all $C_3^2 = 3$ elements commute in pairs.

4) Only elements of 3 “similar” pairs commute, i.e. ω_{11}, ω_{12}; ω_{13}, ω_{14}; ω_{15}, ω_{16}, and the result of multiplication is the complete subset 3). All other non commutative permutations from $C_6^2 = 15$ pairs, form the two-factor products that belong to the subset 5).

5) Permutations of 4 “similar” pairs commute in our aforesaid meaning: ω_{17}, ω_{18}; ω_{19}, ω_{20}; ω_{21}, ω_{22}; ω_{23}, ω_{24}. The remained non commutative pairs from the $C_8^2 = 28$ representatives of the subset 5) form the products whose result is the permutation that belongs either to the subset 3), or 5).

Turning to the interrelations between subsets (2.1), it is easy to notice that the number of commutative in pairs permutations is even less than those which are from one and the same subset.

3. Results

As it was said, the purpose of the given paper is an explicit construction of the covering algebraic equations that are raised by the

L.-L.e. [5], [11]-[13] and whose respective vector boundary Riemann problem (1.1)-(1.5) has non commutative matrix coefficient. So, first of all we briefly introduce some results that concern the construction of the two-sheeted covering of a torus T, and include simultaneously an analytic solution of the L.-L.e. [5], [11]-[13]. As it will be shown later, the study of such simple problem will allow constructing the appropriate coverings with the non commutative monodromy groups (1.1), (1.2).

Hence, the sought for vector function is the following:

$$F(z,u) = \{F_1(z,u),\ F_2(z,u)\} \in h_0(L;T) - ? \qquad (3.0)$$

The well-known algebraic equation of a torus T and its genus ρ are given below:

$$T : u^2 = \prod_{j=1}^{2} (z - a_j)(z - b_j);\ \ \rho = 1. \qquad (3.1)$$

The corresponding homogeneous vector boundary Riemann-Hilbert problem that raises the two-sheeted covering of T and its algebraic equation [1], [2], including an investigation of the L.-L.e. [5], [11], has the following boundary condition [12], [13]:

$$F^+(t,v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} F^-(t,v), \quad (t,v) \in L, \quad D^{-1} \,|\, (F), \qquad (3.2)$$

where:

$$v^2 = \prod_{j=1}^{2} (t - a_j)(t - b_j); \quad L = (a, v\,(\alpha); \beta, v\,(\beta)); \qquad (3.3)$$

D is the two-dimensional vector-divisor of infinities on T, and all other symbols are described in the Section 1.

Geometrically, it means that two copies of T are pasting cross-wise together along the contour-cut L.

Since the matrix coefficient in (3.2) is commutative, there exists the linear transformation S:

$$\exists S = S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Phi(z,u) = S \times F(z,u) \qquad (3.4)$$

that is constant everywhere on L and which takes (3.2) to the equivalent system of homogeneous boundary scalar Riemann-Hilbert problems on T [12]-[14]:

$$\Phi^+(t,v) = \mathrm{diag}\ (-1, 1)\ \Phi^-(t,v), \quad (t,v) \in L,\ \ D^{-1} \,|\, (\Phi). \qquad (3.5)$$

Because of [3], [4], [6], [7] we get the required solutions of (3.5):

$$\Phi_1(z,u) = \Phi_{11}(z,u) \cup \Phi_{12}(z,u);$$

$$\Phi_{11}(z,u) = g_1(z)\chi(z,u),\ \Phi_{12}(z,u) = (h_2(z)+u)\,\chi(z,u),$$

$$\Phi_2(z,u) = 1;\ (z,u) \in T, \tag{3.6}$$

whose functions $g_1(z), h_2(z)$ are explicitly obtained polynomials with the known coefficients and respective powers 1 and 2.

$$\chi(z,u) = \exp\left\{\frac{1}{2}\int_L \mathrm{d}\,w - \int_{(z*,\xi^*)}^{(m_1,\mu_1)} \mathrm{d}\,w + \int_{(z*,\xi^*)}^{(a,v(a))} \mathrm{d}\,w\right\}. \tag{3.7}$$

In (3.7): $\mathrm{d}\,w$ is the gap-like analogy of Cauchy kernel on T; the point (m_1, μ_1) is found by means of the Jacobi conversion problem [1] and is written explicitly in terms of elliptic Jacobi functions [4], [6], [16]; (z^*, ξ^*) is an arbitrary point of T.

Then the written below matrices represent the c.s.m. (while $j=1$) and n.s.m. (while $j=2$) of (3.5), respectively:

$$Y_j(z,u) = \mathrm{diag}\,(\Phi_{1j}(z,u), 1) \quad (j = 1,\,2). \tag{3.8}$$

Taking into account formula (3.4), matrices (3.8) can be transformed into the c.s.m. $(j=1)$ and n.s.m. $(j=2)$ of the original problem (3.0)-(3.3):

$$\mathrm{X}_j(z,u) = \frac{1}{\sqrt{2}}\begin{bmatrix} -\Phi_{1j}(z,u) & 1 \\ \Phi_{1j}(z,u) & 1 \end{bmatrix} \quad (j = 1,\,2). \tag{3.9}$$

In its turn, the sought for vector function from (3.0) looks like:

$$F(z,u) = \begin{bmatrix} F_{1j}(z,u) \\ F_{2j}(z,u) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -\Phi_{1j}(z,u)+1 \\ \Phi_{1j}(z,u)+1 \end{bmatrix} \ (j = 1,\ 2). \tag{3.10}$$

Further, we get two corresponding algebraic equations of the two-sheeted coverings of the torus T, that are given by (3.1):

$$\lambda^2 + c_{1j}(z,u)\lambda + c_{2j}(z,u) = 0; \quad c_{1j}(z,u) = -(F_{1j}(z,u) + F_{2j}(z,u)) = \sqrt{2},$$

$$c_{2j}(z,u) = F_{1j}(z,u) \times F_{2j}(z,u) = (1 - \Phi_{1j}^2(z,u))/2. \tag{3.11}$$

Using the substitution:

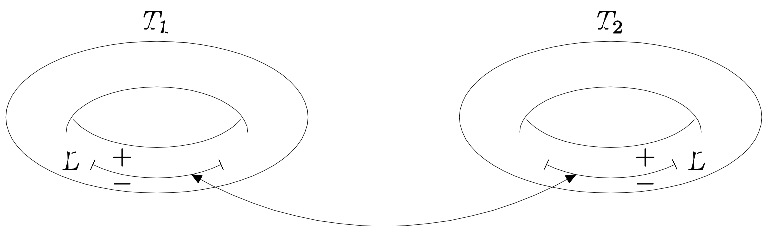
$$W = \sqrt{2}\lambda + 1$$

for λ, we reduce (3.11) to the unified algebraic equation of the two-sheeted covering T^* over the torus T:

$$T^*: \quad W^2 - \Phi_{1j}^2(z,u) = 0 \quad (j = 1, 2). \tag{3.12}$$

The latter includes both equations (3.11) and is in conformity with other known results [5], [12], [13].

Geometrically, the covering T^* over T is shown here as the crosswise pasting of two torus' copies along the crosswise cut of the contour L from (3.3).



$$\begin{aligned} &\rho^* = 2. \\ &L = (\alpha, v(\alpha); \beta, v(\beta)): \quad (12). \end{aligned} \tag{3.13}$$

In (3.13), the commutative monodromy group is given on the torus T, and its relevant permutation from (1.2) is (12) with the appropriate ramification points $(\alpha, v(\alpha))$; $(\beta, v(\beta))$ that are the endpoints of the open contour L.

As it is well known [17], the genus ρ^* of the covering R^* over the compact / algebraic Riemann surface R can be computed by the following formula:

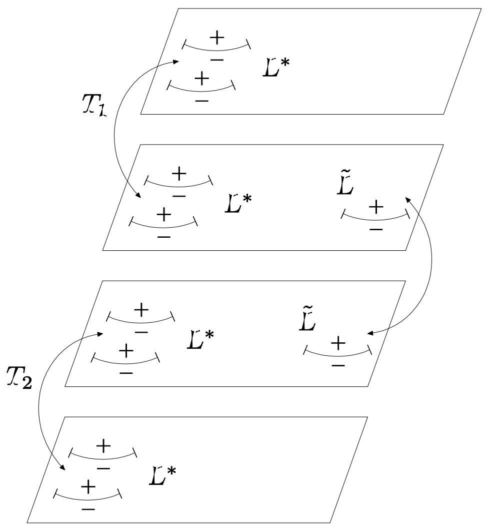
$$\rho^* - 1 = m\,(\rho - 1) + \frac{1}{2}V, \tag{3.14}$$

where: ρ is the genus of the initial surface R; m is the dimension of the permutations from the monodromy group that is fixed on R; $V = \sum_{k=1}^{p}(q_k - 1)$ is the ramification index of the covering R^*, whose item q_k in the sum symbol is the order of the respective kth ramification point, and all of these p points are summed as many times, as their multiplicities are considered.

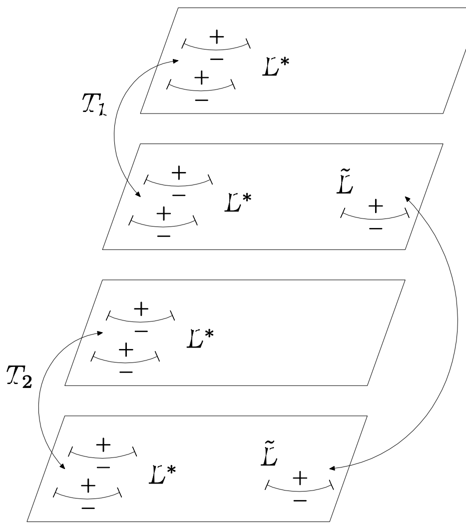
In (3.13), ρ^* is computed by the formula (3.14), and the monodromy group of T^* is written according to (1.1), (1.2).

Using the results of [3], [4] we can assert that the original vector problem (3.0)-(3.3) on T with the second-order commutative permutation matrix coefficient generates several other homogeneous vector boundary Riemann-Hilbert problems on the complex plane **C** with the fourth-order non commutative permutation matrix coefficients. An explicit solution of such problems and construction of the algebraic equations of the respective coverings are done by means of the general method and technique of [3], [4].

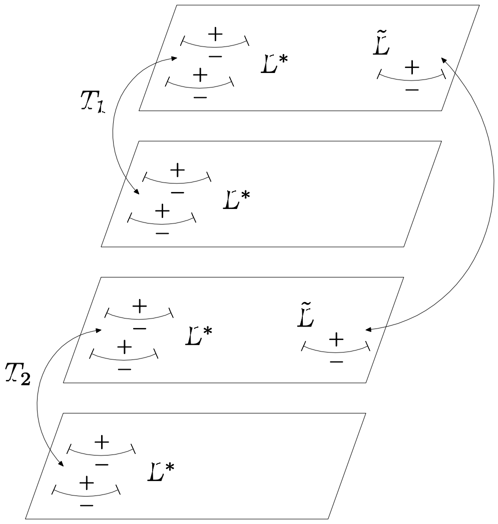
Since a torus is the two-sheeted covering of the complex plane **C,** then the surface from (3.13) is the four-sheeted covering over **C**. Turning again to the results of [3], [4] which describe the non commutative monodromy groups generated by the commutative ones that are given on the surfaces with the bigger genera, we can introduce geometrically the new coverings R^*_μ $(\mu = \overline{1, 4})$ over **C**. These surfaces are evoked by (3.13), are equivalent to T^* topologically, but not conformally, and their monodromy groups are non commutative. So, the aforesaid coverings R^*_μ $(\mu = \overline{1, 4})$ geometrically look like:



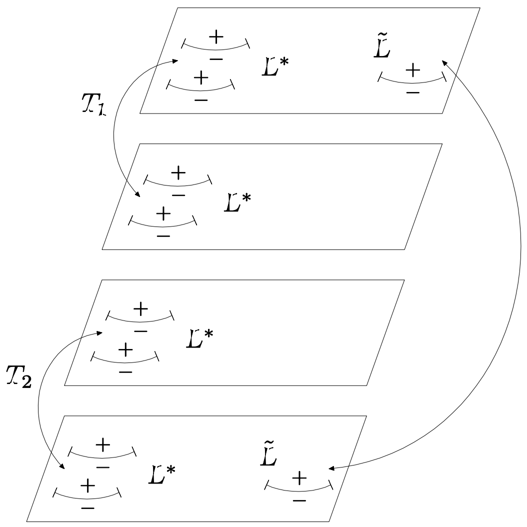
$$\begin{aligned} &L^* = \bigcup_{j=1}^{2} (a_j,\ b_j): \quad (12)(34) \\ &\tilde{L} = (\alpha,\ \beta): \quad (1)(23)(4);\ \rho_1^* = 2. \end{aligned} \tag{3.15}$$



$$L^* = \bigcup_{j=1}^{2}(a_j,\, b_j): \quad (12)(34)$$
$$\widetilde{L} = (\alpha,\ \beta): \quad (1)(24)(3);\ \rho_2^* = 2. \tag{3.16}$$



$$L^* = \bigcup_{j=1}^{2}(a_j,\, b_j): \quad (12)(34)$$
$$\widetilde{L} = (\alpha,\ \beta): \quad (13)(2)(4);\ \rho_3^* = 2. \tag{3.17}$$



$$L^* = \bigcup_{j=1}^{2}(a_j,\ b_j): \quad (12)(34)$$
$$\tilde{L} = (\alpha,\ \beta): \quad (14)(2)(3);\ \rho_4^* = 2. \qquad (3.18)$$

Hence, according to (1.1)-(1.5) and (3.15)-(3.18) the respective vector boundary Riemann-Hilbert problems on **C** have the non commutative matrix coefficients, the unknown vector-valued functions are the following:

$$f(z) = \{f_j(z)\}_{j=1}^{4} \in h_0(L^* \cup \tilde{L}; \mathbf{C})\ \textbf{- ?} \qquad (3.19)$$

and satisfy the boundary condition:

$$f^+(t) = \left(\sum_{i=1}^{2} M_i \delta(t; L_i)\right) f^-(t), \quad t \in L_1 \cup L_2, \quad L_1 = L^*, \quad L_2 = \tilde{L}. \qquad (3.20)$$

In (3.20), $M_i\ (i = 1,\ 2)$ are the permutation matrices that are raised by the corresponding permutations from (3.15)-(3.18) non commutative monodromy groups and look as:

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{or} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.21}$$

It should be noted again that the problem (3.19)-(3.21) is equivalent geometrically to the cross-wise pasting of two copies of the torus T along the contour-cut L from the problem (3.0)-(3.3). This fact is true in the topological, but not in the conformal meaning. In the considered case of (3.19)-(3.21), the required pasting together construction is done not as the two-sheeted covering of the torus T but in terms of the four-sheeted covering of **C** and represents the above mentioned surfaces R_μ^* $(\mu = \overline{1,4})$.

The normal basis of R_μ^* $(\mu = \overline{1,4})$, in accordance with [2]-[4], [6] is written below:

$$\{\Phi_{11}(z,u), \Phi_{12}(z,u), u, 1\}, \tag{3.22}$$

where the functions $\Phi_{1j}(z,u)$ $(j = 1, 2)$ are from (3.6), (3.7), and u is described by the equation (3.1).

If r_j $(j = \overline{1,3})$ are the respective orders of the functions $\{\Phi_{11}(z,u), \Phi_{12}(z,u), u\}$ at infinity, then $\sum_{j=1}^{3}(r_j - 1) = 2$ and equals the genus ρ^* of the four-sheeted covering R_μ^* $(\mu = \overline{1,4})$. This fact is in conformity with the well-known Riemann-Roch theorem [1], [17] as one of its corollaries.

The sought for c.s.m. of the problem (3.19)-(3.21) is got due to the technique of [3], [4] and c.s.m., n.s.m. (3.9) that were obtained solving the problem (3.0)-(3.3):

$$X(z) = \begin{bmatrix} \Phi_{11}(z,u) & \Phi_{12}(z,u) & u & 1 \\ \Phi_{11}(z,-u) & \Phi_{12}(z,-u) & -u & 1 \\ -\Phi_{11}(z,-u) & -\Phi_{12}(z,-u) & -u & 1 \\ -\Phi_{11}(z,u) & -\Phi_{12}(z,u) & u & 1 \end{bmatrix}, \tag{3.23}$$

$$
\mathrm{X}(z)=\begin{bmatrix} \Phi_{11}(z,u) & \Phi_{12}(z,u) & u & 1 \\ \Phi_{11}(z,-u) & \Phi_{12}(z,-u) & -u & 1 \\ -\Phi_{11}(z,u) & -\Phi_{12}(z,u) & u & 1 \\ -\Phi_{11}(z,-u) & -\Phi_{12}(z,-u) & -u & 1 \end{bmatrix}. \qquad (3.24)
$$

Matrices (3.23), (3.24) are the c.s.m. for the first and the fourth, the second and third values of M_2 from (3.21) respectively.

Results (3.22)-(3.24) are obtained as the particular simplest case of [4]. Taking into consideration formulae (3.23), (3.24) we construct the sought for algebraic equations [15] of the coverings R_μ^* $(\mu=\overline{1,4})$ whose geometrical pictures are (3.15)-(3.18):

$$
R_\mu^* \ (\mu=\overline{1,4}):\ V^4+d_1(z)V^3+d_2(z)V^2+d_3(z)V+d_4(z)=0,
$$

$$
\Phi_{1j\pm}=\Phi_{1j}(z,\pm u)\ \ (j=1,2);\ \ d_1(z)=-4;
$$

$$
d_2(z)=4+2(1-u^2)-(\Phi_{11+}+\Phi_{12+})^2-(\Phi_{11-}+\Phi_{12-})^2;
$$

$$
d_3(z)=2(2(u^2-1)+(1+u)(\Phi_{11-}+\Phi_{12-})^2+(1-u)(\Phi_{11+}+\Phi_{12+})^2); \quad (3.25)
$$

$$
d_4(z)=((u+1)^2-(\Phi_{11+}+\Phi_{12+})^2)((u-1)^2-(\Phi_{11-}+\Phi_{12-})^2).
$$

Thus, we have obtained the required algebraic equation (3.25) that appears as unified for all coverings R_μ^* $(\mu=\overline{1,4})$.

In comparison with the joint formula (3.12) that unites the pair of different algebraic equations for the surface T^*, the unified equation (3.25) for the topologically equivalent coverings R_μ^* $(\mu=\overline{1,4})$ is unique for all four cases. It is quite natural, though (3.25) is generated by (3.12). Really, R_μ^* $(\mu=\overline{1,4})$ cover the complex plane, and the corresponding vector boundary Riemann-Hilbert problem (3.19)-(3.21) has no n.s.m., but only c.s.m. [2], [4]. This completely explains uniqueness of the existing algebraic equation (3.25) [2], [4].

Therefore, we have constructed explicitly the unknown algebraic equations of the surfaces that are raised by the two-sheeted covering of a torus (3.12) from the solution of L.-L.e. [5], [12], [13]. Moreover, the monodromy groups of these surfaces are non commutative. It means that that the originally given problem is solved completely and the goal of the present article is attained.

4. Remarks and conclusions

Though the suggested here problem is solved explicitly, it is clear that the given group of permutations S_4 is not studied completely even in the meaning of commutativity in pairs and over the complex plane. It implies that not all possible existing multi-valued functions with non commutative in pairs monodromy groups from S_4 are investigated yet. In its turn, the relevant vector fields are not constructed on **C**, at least.

Further, if commutativity in bigger sets than pairs is required, the problem of the monodromy group investigation becomes almost unobservable. Hence, the future analytical research of S_4 remains urgent as from the pure mathematical viewpoint, as from the applied industrial and engineering.

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