THE SIMPLE CHAOTIC GENERAL ECONOMIC EQUILIBRIUM GROWTH MODEL

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Abstract. *The basic aim of this paper is to construct a relatively simple chaotic general economic equilibrium growth model that is capable of generating stable equilibriums, cycles, or chaos.*
A key hypothesis of this work is based on the idea that the coefficient $\pi = \dfrac{db\, m_{RT}}{(\alpha - 1)\, m_{RS}}$ *plays a crucial role in explaining local stability of the general economic equilibrium output, where, b – the coefficient of the demand function, d – the coefficient of the marginal cost function,* m_{RS} *– the marginal rate of substitution,* m_{RT} *– marginal rate of transformation,* α *- the coefficient of marginal cost growth.*

Keywords: *General equilibrium, Output, Chaos, logistic equation.*

1. Introduction

Chaos theory reveals structure in unpredictable dynamic systems. It is important to construct deterministic, nonlinear general economic equilibrium growth model that elucidate irregular, unpredictable general economic equilibrium behaviour.

Chaos theory can explain effectively unpredictable economic long time behaviour arising in a deterministic dynamical system because of sensitivity to initial conditions. A deterministic dynamical system is perfectly predictable given perfect knowledge of the initial condition, and is in practice always predictable in the short term. The key to long-term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions.

Chaos theory started with Lorenz's (1963) discovery of complex dynamics arising from three nonlinear differential equations leading to turbulence in the weather system. Li and Yorke (1975) discovered that the

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simple logistic curves can exhibit very complex behaviour. Further, May (1976) described chaos in population biology. Chaos theory has been applied in economics by Benhabib and Day (1981, 1982), Day (1982, 1983, 1997), Gandolfo (2009), Grandmont (1985), Goodwin (1990), Medio (1993, 1996), Lorenz (1993), Jablanovic (2010, 2011a, 2011b, 2011c, 2012), among many others.

The basic aim of this paper is to provide a relatively simple chaotic general economic equilibrium output growth model that is capable of generating stable equilibriums, cycles, or chaos.

2. General economic equilibrium

Marginal rate of substitution, m_{RS}, is the amount of a good *A* that a consumer is willing to give up in order to obtain one additional unit of good *B*. When output markets are perfectly competitive, all consumers allocate their budgets so their marginal rates of substitution between two goods are equal to the price ratio. For our two goods, *A*, and *B:*

$$m_{RS} = P_A / P_B \tag{1}$$

where m_{RS} – marginal rate of substitution, P_A – the price of good *A*, P_B – the price of good *B*.

Each profit-maximizing firm will produce its output up to the point at which price is equal to marginal cost. For goods *A* and *B*:

$$P_A = MC_A \text{ and } P_B = MC_B. \tag{2}$$

Marginal rate of transformation is an amount of good *A* that must be given up to produce one additional unit of a good *B*. The marginal rate of transformation is the ratio of the marginal cost of producing good *A*, MC_A, to the marginal cost of producing good *B*, MC_B, or:

$$m_{RT} = MC_A / MC_B \tag{3}$$

where m_{RT} – marginal rate of transformation, MC_A – the marginal cost of poducing good *A*, MC_B – the marginal cost of producing good *B*.

An economy produces output efficiently only if, for each consumer:

$$m_{RS} = m_{RT} \tag{4}$$

where m_{RS} – the marginal rate of substitution, m_{RT} – marginal rate of transformation.

The efficient combination of outputs is produced when the marginal rate of transformation between the two goods (which measures the cost of producing one good relative to the other) is equal to the consumer's marginal rate of substitution (which measures the marginal benefit of consuming one good relative to the other).

Because the marginal rate of transformation is equal to the ratio of the marginal costs of production, then:

$$m_{RT} = MC_A / MC_B = P_A / P_B = m_{RS} \quad (5)$$

when output and input markets are competitive, production will be efficient in that the m_{RT} is equal to the m_{RS}.

In a competitive output market, consumers consume to the point where their marginal rate of substitution is equal to the price ratio. Producers choose outputs so that the marginal rate of transformation is equal to the price ratio. Because the m_{RS} equals the m_{RT}, the competitive output market is efficient. Any other price ratio will lead to an excess demand for good *A* and an excess supply of the good *B*.

Suppose the demand function is:

$$Q_t = a - bP_t \quad (6)$$

where *P* – the price; *Q* – the demanded quantity; *a, b* – the coefficients of the demand function.

Further, suppose the quadratic marginal-cost functions (MC_A and MC_B) have the identical form:

$$MC_t = c + dQ_t + eQ_t^2 \quad (7)$$

where *MC* – marginal cost; *Q* – output; *c, d*, and *e* – coefficients of the quadratic marginal-cost function.

Further, it is supposed that:

$$MC_{A,\, t+1} = MC_{A,\, t} + \alpha\, MC_{A,\, t+1} \quad (8)$$

or:

$$(1 - \alpha)MC_{A,\, t+1} = MC_{A,t}. \quad (9)$$

Substitution (2) and (5) in (9) gives:

$$(1 - \alpha)m_{RS}P_{B,\, t+1} = m_{RT}\, MC_{B,t}. \quad (10)$$

Firstly, it is supposed that $a = 0$ and $c = 0$. Further, by substitution (6) and (7) in (10) one derives:

$$Q_{B,t+1} = \frac{dbm_{RT}}{(\alpha - 1)\, m_{RS}} Q_{B,t} - \frac{ebm_{RT}}{(1 - \alpha)\, m_{RS}} Q_{B,t}^{\ 2}. \quad (11)$$

Further, it is assumed that the equilibrium output, $Q_{B,t}$, is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the equilibrium output growth rate depends on the current size of the equilibrium output, $Q_{B,t}$, relative to its maximal size in its time series $Q^m_{B\,t}$. We introduce $q_{B\,t}$ as $q_{B,\,t} = Q_{B,\,t}/Q^m_{B\,t}$. Thus $q_{B,t}$ range is between 0 and 1. Again we index $q_{B,t}$ by t, i.e., write $q_{B,t}$ to refer to the size at time steps t = 0,1,2,3,... Now, growth rate of the equilibrium output is measured as:

$$q_{B,t+1} = \frac{db\, m_{RT}}{(\alpha - 1)\, m_{RS}} q_{B,t} - \frac{ebm_{RT}}{(1-\alpha)\, m_{RS}} q_{B,t}{}^2. \tag{12}$$

This model given by equation (12) is called the logistic model. For most choices of b, d, e, m_{RS}, m_{RT}, and α there is no explicit solution for (12). Namely, knowing b, d, e, m_{RS}, m_{RT}, and α and measuring $q_{B,0}$ would not suffice to predict $q_{B,t}$ for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect – the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

This kind of difference equation (12) can lead to very interesting dynamic behaviour, such as cycles that repeat themselves every two or more periods, and even chaos, in which there is no apparent regularity in the behaviour of $q_{B,t}$. This difference equation (12) will possess a chaotic region. Two properties of the chaotic solution are important: firstly, given a starting point $q_{B,0}$ the solution is highly sensitive to variations of the parameters b, d, e, m_{RS}, m_{RT}, and α; secondly, given the parameters b, d, e, m_{RS}, m_{RT}, and α the solution is highly sensitive to variations of the initial point $q_{B,0}$. In both cases the two solutions are for the first few periods rather close to each other, but later on they behave in a chaotic manner.

3. Logistic Equation and general equilibrium output

The logistic map is often cited as an example of how complex, *chaotic* behaviour can arise from very simple *non-linear* dynamical equations. The logistic model was originally introduced as a *demographic model* by *Pierre François Verhulst*. It is possible to show that iteration process for the logistic equation:

$$z_{t+1} = \pi z_t(1 - z_t),\ \pi \in [0,4],\ z_t \in [0,1] \tag{13}$$

is equivalent to the iteration of growth model (12) when we use the following identification:

$$z_t = \frac{e(\alpha - 1)}{d(1-\alpha)} q_{B,t} \text{ and } \pi = \frac{dbm_{RT}}{(\alpha - 1)m_{RS}}. \tag{14}$$

Using (12) and (14) we obtain:

$$z_{t+1} = \left[\frac{e\,(\alpha - 1)}{d\,(1-\alpha)}\right] q_{BA,\,t+1} =$$

$$= \left[\frac{e(\alpha - 1)}{d(1-\alpha)}\right]\left\{\left[\frac{dbm_{RT}}{(\alpha - 1)\,m_{RS}}\right] q_{B,t} - \left[\frac{ebm_{RT}}{(1-\alpha)\,m_{RS}}\right] q_{B,t}{}^2\right\} =$$

$$= \left[\frac{ebm_{RT}}{(1-\alpha)m_{RS}}\right] q_{B,t} - \left[\frac{e^2 b\,(\alpha - 1)m_{RT}}{d(1-\alpha)^2 m_{RS}}\right] q_{B,t}{}^2.$$

On the other hand, using (13), and (14) we obtain:

$$z_{t+1} = \pi z_t(1 - z_t) = \left[\frac{bdm_{RT}}{(\alpha - 1)m_{RS}}\right]\left[\frac{e\,(\alpha - 1)}{d(1-\alpha)}\right] q_{B,t}\left\{1 - \left[\frac{e(\alpha - 1)}{d(1-\alpha)}\right] q_{B,t}\right\} =$$

$$= \left[\frac{ebm_{RT}}{(1-\alpha)m_{RS}}\right] q_{B,t} - \left[\frac{e^2 b\,(\alpha - 1)\,m_{RT}}{d(1-\alpha)^2\,m_{RS}}\right] q_{B,t}^2.$$

Thus we have that iterating $q_{B,t+1} = \frac{db_{RT}}{(\alpha - 1)m_{RS}} q_{B,t} - \frac{ebm_{RT}}{(1-\alpha)m_{RS}} q_{B,\,t}^2$ is really the same as iterating $z_{t+1} = \pi z_t(1 - z_t)$ using $z_t = \frac{e\,(\alpha - 1)}{d\,(1-\alpha)} q_{B,t}$ and:

$$\pi = \frac{dbm_{RT}}{(\alpha - 1)\,m_{RS}}.$$

It is important because the dynamic properties of the logistic equation (13) have been widely analyzed (Li and Yorke (1975), May (1976)).

It is obtained that:

(i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$;

(ii) For $1 < \pi < 3{,}57$ there exist fixed points the number of which depends on π;

(iii) For $1 < \pi < 2$ all solutions monotonically increase to $z = (\pi - 1)/\pi$;

(iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1)/\pi$;

(v) For $3 < \pi < 4$ all solutions will continuously fluctuate;

(vi) For $3{,}57 < \pi < 4$ the solution become "chaotic" which means that there exist totally a periodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever.

4. Conclusion

This paper suggests conclusion for the use of the simple chaotic general equilibrium growth model in predicting the fluctuations of the equilibrium output. The model (12) has to rely on specified parameters *b, d, e,* m_{RS}, m_{RT}, and α and initial value of the equilibrium output, q_{B0}. But even slight deviations from the values of parameters *b, d, e,* m_{RS}, m_{RT}, and α and initial value of the equilibrium output show the difficulty of predicting a long-term behaviour of the general equilibrium output.

A key hypothesis of this work is based on the idea that the coefficient $\pi = \dfrac{db\, m_{RT}}{(\alpha - 1)\, m_{RS}}$ plays a crucial role in explaining local stability of the general equilibrium output, where, *b* – the coefficient of the demand function *d* – the coefficient of the marginal cost function, m_{RS} – the marginal rate of substitution, m_{RT} – marginal rate of transformation, α – the coefficient of marginal cost growth.

REFERENCES

[1] Benhabib, J., & Day, R. H. (1981), *Rational choice and erratic behaviour,* Review of Economic Studies, **48,** 459-471.

[2] Benhabib, J., & Day, R. H. (1982), *Characterization of erratic dynamics in the overlapping generation model*, Journal of Economic Dynamics and Control, **4,** 37-55.

[3] Benhabib, J., & Nishimura, K. (1985), *Competitive equilibrium cycles,* Journal of Economic Theory, **35,** 284-306.

[4] Day, R. H. (1982), *Irregular growth cycles*, American Economic Review, **72,** 406-414.

[5] Day, R. H. (1983), *The emergence of chaos from classical economic growth*, Quarterly Journal of Economics, **98,** 200-213.

[6] Day, R. H. (1997), *Complex economic dynamics volume I: An introduction to dynamical systems and market mechanism,* In Discrete Dynamics in Nature and Society (pp. 177-178), 1. MIT Press.

[7] Gandolfo, G. (2009), *Economic dynamics* (4th ed.), Berlin: Springer-Verlag.

[8] Goodwin, R. M. (1990), *Chaotic economic dynamics,* Oxford: Clarendon Press.

[9] Grandmont, J. M. (1985), *On endogenous competitive business cycles*, Econometrica, **53,** 994-1045.
[10] Jablanović, V. (2010), *Chaotic population growth,* Belgrade, Cigoja.
[11] Jablanović, V. (2011a), *The chaotic saving growth model: G 7,* Chinese Business Review, **10**(5), 317-327.
[12] Jablanović, V. (2011b), *The chaotic Monopoly Price Growth Model*. Chinese Business Review, **10**(11), 985-990.
[13] Jablanović, V. (2011c), *The Chaotic Production Growth Model of the Monopoly Firm,* Hyperion International Journal of Econophysics & New Economy, Volume **4,** Issue 2, 2011, pp. 285-292.
[14] Jablanović, V. (2012), *Budget Deficit and Chaotic Economic Growth Models,* Aracne editrice, S.R.L. Roma
[15] Li, T., & Yorke, J. (1975), *Period three implies chaos*, American Mathematical Monthly, **8,** 985-992.
[16] Lorenz, E. N. (1963), *Deterministic nonperiodic flow*, Journal of Atmospheric Sciences, **20,** 130-141.
[17] Lorenz, H. W. (1993), *Nonlinear dynamical economics and chaotic motion* (2nd ed.). Heidelberg: Springer-Verlag.
[18] May, R. M. (1976), *Mathematical models with very complicated dynamics*, Nature, **261,** 459-467.
[19] Medio, A. (1993), *Chaotic dynamics: Theory and applications to economics,* Cambridge: Cambridge University Press.
[20] Medio, A. (1996), *Chaotic dynamics. Theory and applications to economics*, Cambridge University Press, In De Economist, **144**(4), 695-698.
[21] Peitgen, H-O., Jürgens, H., & Saupe, D. (1992), *Chaos and fractals-new frontiers of science,* New York: Springer-Verlag.
[22] Rössler, O. E. (1976), *An equation for continuous chaos,* Physics Letters A, **57,** 397-398, Tu, P. N. V. (1994), *Dynamical systems,* Verlag: Springer.