THE CHAOTIC PRODUCTION GROWTH MODEL OF THE MONOPOLY FIRM

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Abstract. *Chaos theory is used to prove that erratic and chaotic fluctuations can indeed arise in completely deterministic models. Chaos theory reveals structure in aperiodic, dynamic systems. Chaotic systems exhibit a sensitive dependence on initial conditions. Seemingly insignificant changes in the initial conditions produce large differences in outcomes. This is very different from stable dynamic systems.*

To maximize profit, the monopolist must first determine its costs and the characteristics of market demand. Given this knowledge, the monopoly firm must then decide how much to produce. The monopoly firm can determine price, and the quantity it will sell at that price follows from the market demand curve.

The basic aim of this paper is to construct a relatively simple chaotic growth model of the monopoly quantity that is capable of generating stable equilibria, cycles, or chaos.

A key hypothesis of this work is based on the idea that the coefficient $\pi = \left[\frac{e\,b}{m(\alpha - 1)(e+1)} \right]$*, plays a crucial role in explaining local stability of the monopoly's production , where, b – the coefficient of the marginal cost function of the monopoly firm, m – the coefficient of the inverse demand function, e – the coefficient of the price elasticity of the monopoly's demand,* α *– the coefficient.*

Keywords: *Monopoly, Production, Chaos, Chaotic model, Logistic Equation.*

1. Introduction

Chaos theory attempts to reveal structure in nonlinear, unpredictable dynamic systems. It is important to construct **deterministic**, nonlinear economic dynamic models that elucidate irregular, unpredictable economic behavior. Deterministic chaos refers to irregular or chaotic motion that is generated by nonlinear systems evolving according to dynamical laws that uniquely determine the state of the system at all times from a knowledge of the system's previous history. Chaos embodies three important principles:

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(i) extreme sensitivity to initial conditions; (ii) cause and effect are not proportional; and (iii) nonlinearity.

Chaos theory can explain effectively unpredictable economic long time behavior arising in a deterministic dynamical system because of sensitivity to initial conditions. It must be emphasized that a deterministic dynamical system is perfectly predictable given perfect knowledge of the initial condition, and is in practice always predictable in the short term. The key to long-term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions.

Chaos theory started with Lorenz's (1963) discovery of complex dynamics arising from three nonlinear differential equations leading to turbulence in the weather system. Li and Yorke (1975) discovered that the simple logistic curve can exibit very complex behaviour. Further, May (1976) described chaos in population biology. Chaos theory has been applied in economics by Benhabib and Day (1981, 1982), Day (1982, 1983, 1997), Grandmont (1985), Goodwin (1990), Medio (1993, 1996), Lorenz (1993), among many others.

The basic aim of this paper is to provide a relatively simple chaotic growth model of the monopoly's production that is capable of generating stable equilibria, cycles, or chaos.

2. A Simple Chaotic Model of a Profit-Maximizing Monopoly

In the model of a profit-maximizing monopoly, take the inverse demand function:

$$P_t = n - mQ_t. \tag{1}$$

Where P – monopoly price; Q – monopoly output, n, m – coefficients of the inverse demand function.

Further, suppose the quadratic marginal-cost function for a monopoly is:

$$MC_t = a + bQ_t + cQ_t^2 \tag{2}$$

MC – marginal cost; Q – monopoly output; a,b,c – coefficients of the quadratic marginal-cost function.

Marginal revenue is:

$$MR_t = P_t\left[1+\left(\frac{1}{e}\right)\right] \tag{3}$$

MR – marginal revenue; *P* – monopoly price; *e* – the coefficient of the price elasticity of demand.

Thus the profit-maximizing condition is that:

$$MR_t = MC_t. \quad (4)$$

Further,

$$MR_{t+1} = MR_t + \Delta\, MR. \quad (5)$$

Or

$$MR_{t+1} = MR_t + \alpha\, MR_{t+1}, \quad (6)$$

i.e.

$$(1-\alpha)\, MR_{t+1} = MR_t. \quad (7)$$

Thus, the chaotic model of profit-maximizing monopoly is presented by the following equations:

$$(1-\alpha)\, MR_{t+1} = MR_t$$

$$MR_t = MC_t \ (4)$$

$$MR_t = P_t\left[1 + \left(\frac{1}{e}\right)\right]$$

$$MC_t = a + b\, Q_t + c\, Q_t^{\ 2}$$

$$P_{t+1} = n - m\, Q_{t+}$$

where: *Q* – output of the monopolist; *MC* – marginal cost; *MR* – marginal revenue; *P* – monopoly price; *e* – the coefficient of the price elasticity of demand; *n*, *m* – coefficients of the inverse demand function; *a,b,c* – coefficients of the quadratic marginal-cost function.

Firstly, it is supposed that $a = 0$ and $n = 0$. By substitution one derives:

$$Q_{t+1} = \frac{e\,b}{m\,(\alpha-1)\,(e+1)}\,Q_t + \frac{e\,c}{m\,(\alpha-1)\,(e+1)}\,Q_t^{\ 2}. \quad (8)$$

Further, it is assumed that the monopoly's output is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the monopolist's output growth rate depends on the current size of the monopolist's output, *Q*, relative to its maximal size in its time series Q^m. We introduce *q* as $q = Q/Q^m$. Thus *q* range between 0 and 1. Again we index *q* by *t*, i.e., write q_t to refer to the size at time steps $t = 0,1,2,3,...$ Now growth rate of the monopolist's output is measured as:

$$q_{t+1} = \frac{e\,b}{m\,(\alpha-1)\,(e+1)}\,q_t + \frac{e\,c}{m\,(\alpha-1)\,(e+1)}\,q_t^{\ 2}. \quad (9)$$

This model given by equation (9) is called the logistic model. For most choices of *b*, *c*, *m*, and *e* there is no explicit solution for (9). Namely, knowing *b*, *c*, *m*, and *e* and measuring q_0 would not suffice to predict q_t for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect – the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

This kind of difference equation (9) can lead to very interesting dynamic behavior, such as cycles that repeat themselves every two or more periods, and even chaos, in which there is no apparent regularity in the behavior of q_t. This difference equation (9) will posses a chaotic region. Two properties of the chaotic solution are important: firstly, given a starting point q_0 the solution is highly sensitive to variations of the parameters *b*, *c*, *m*, and *e*; secondly, given the parameters *b*, *c*, *m*, and *e* the solution is highly sensitive to variations of the initial point q_0. In both cases the two solutions are for the first few periods rather close to each other, but later on they behave in a chaotic manner.

3. Logistic Equation

The logistic map is often cited as an example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations. The map was popularized in a seminal 1976 paper by the biologist Robert May. The logistic model was originally introduced as a demographic model by Pierre François Verhulst. The logistic map is an example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations. The map was popularized in a seminal 1976 paper by the biologist Robert May. However, the logistic model was originally introduced as a demographic model by Pierre François Verhulst.

It is possible to show that iteration process for the logistic equation:

$$z_{t+1} = \pi z_t (1 - z_t),\ \pi \in [0, 4],\ z_t \in [0, 1] \tag{10}$$

is equivalent to the iteration of growth model (9) when we use the following identification:

$$z_t = -\frac{e\,c}{e\,b} q_t \tag{11}$$

and

$$\pi = \left[\frac{e\,b}{m(\alpha - 1)(e + 1)} \right] . \tag{12}$$

Using (10), (11) and (12) we obtain:

$$z_{t+1} = -\left(\frac{e\,c}{e\,b}\right)q_{t+1} = -\left(\frac{e\,c}{e\,b}\right)\left[\left(\frac{e\,b}{m\,(\alpha-1)\,(e+1)}\right)q_t + \left(\frac{e\,c}{m\,(\alpha-1)\,(e+1)}\right)q_t^{\,2}\right] =$$

$$= -\left(\frac{ec}{m\,(\alpha-1)\,(e+1)}\right)q_t - \left(\frac{e^2c^2}{ebm\,(\alpha-1)\,(e+1)}\right)q_t^{\,2}.$$

On the other hand, using (10), (11), and (12) we obtain:

$$z_{t+1} = \pi z_t\,(1-z_t) = -\left(\frac{e\,b}{m\,(\alpha-1)\,(e+1)}\right)\left(\frac{e\,c}{e\,b}\right)q_t\left[1+\left(\frac{e\,c}{e\,b}\right)q_t\right]$$

$$= -\left(\frac{e\,c}{m\,(\alpha-1)\,(e+1)}\right)q_t - \left(\frac{e^2\,c^2}{e\,bm\,(\alpha-1)\,(e+1)}\right)q_t^{\,2}.$$

Thus we have that iterating $q_{t+1} = \frac{e\,b}{m\,(\alpha-1)\,(e+1)}q_t + \frac{e\,c}{m\,(\alpha-1)\,(e+1)}q_t^{\,2}$ is really the same as iterating $z_{t+1} = \pi z_t\,(1- z_t)$ using $z_t = -\frac{e\,c}{e\,b}q_t$ and $\pi = \left[\frac{e\,b}{m(\alpha-1)(e+1)}\right]$.

It is important because the dynamic properties of the logistic equation (10) have been widely analyzed (Li and Yorke (1975), May (1976)).

It is obtained that:

(i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$;
(ii) For $1 < \pi < 3{,}57$ there exist fixed points the number of which depends on π;
(iii) For $1 < \pi < 2$ all solutions monotnically increase to $z = (\pi - 1) / \pi$;
(iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1) / \pi$;
(v) For $3 < \pi < 4$ all solutions will continously fluctuate;
(vi) For $3{,}57 < \pi < 4$ the solution become chaotic wihch means that there exist totally aperiodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever.

4. Conclusion

This paper suggests conclusion for the use of the simple chaotic model of a profit – maximizing monopoly in predicting the fluctuations of the monopoly's output. The model (9) has to rely on specified parameters parameters *b*, *c*, *m*, and *e*, and initial value of the monopolist's output, q_0. But even slight deviations from the values of parameters parameters *b*, *c*, *m*, and *e* and initial value of the monopolist's output, show the difficulty of predicting a long-term behavior of the monopoly's output, q_0.

A key hypothesis of this work is based on the idea that the coefficient $\pi = \left[\frac{e\,b}{m(\alpha - 1)(e+1)} \right]$, plays a crucial role in explaining local stability of the monopoly's output, where, *b* – the coefficient of the marginal cost function of the monopoly firm, *m* – the coefficient of the inverse demand function, *e* – the coefficient of the price elasticity of monopoly's demand, α – the coefficient.

The quadratic form of the marginal cost function of the monopoly's is important ingredient of the presented chaotic monopoly's output growth model (9).

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