A CHAOTIC POPULATION GROWTH MODEL AND DISABILITY

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Abstract. *Chaos theory, as a set of ideas, attempts to reveal structure in aperiodic, unpredictable dynamic systems such as the fluctuation of populations. Although chaotic systems can be described by mathematical equations, chaos theory shows the difficulty of predicting their long-range behavior. In this sense, it is important to construct deterministic, nonlinear population dynamic growth model. Chaos embodies three important principles: (i) extreme sensitivity to initial conditions; (ii) cause and effect are not proportional; and (iii) nonlinearity. The basic aim of this paper is to provide a relatively simple chaotic population growth model that is capable of generating stable equilibria, cycles, or chaos depending on parameter values including the problem of disability.*

A key hypothesis of this work is based on the idea that the coefficient $\pi = \frac{p}{p - m\,h\,f}$ *plays a crucial role in explaining local stability of the population, where f – the share of population which belongs to labour force; m – the coefficient of marginal productivity of disabled memebers of society; p – the real gross domestic product per capita; h – the share of labour force which belong to disable people. The estimated chaotic population model shows the stable and declining population growth in the EU-27 countries in the observed period.*

Keywords: *chaos, population growth, disability, EU-27.*

1. Introduction

Chaos theory is used to prove that erratic and chaotic fluctuations can indeed arise in completely deterministic models. Chaos theory reveals structure in aperiodic, dynamic systems. The number of nonlinear business cycle models use chaos theory to explain complex motion of the economy. Chaotic systems exhibit a sensitive dependence on initial conditions: seemingly insignificant changes in the initial conditions produce large differences in outcomes. This is very different from stable dynamic

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systems in which a small change in one variable produces a small and easily quantifiable systematic change.

Chaos theory started with Lorenz's (1963) discovery of complex dynamics arising from three nonlinear differential equations leading to turbulence in the weather system. Li and Yorke (1975) discovered that the simple logistic curve can exibit very complex behaviour. Further, May (1976) described chaos in population biology. Chaos theory has been applied in economics by Benhabib and Day (1981, 1982), Day (1982, 1983, 1997), Grandmont (1985), Goodwin (1990), Medio (1993, 1996), Medio, A. and Lines, M. (2004), Lorenz (1993), Shone, R. (1999) among many others [1÷20].

The basic aim of this paper is to provide a relatively simple chaotic population growth model that is capable of generating stable equilibria, cycles, or chaos depending on parameter values.

2. The chaotic population growth model and disability

Irregular population movement can be analyzed in formal framework of the chaotic growth model:

$$m = \frac{\Delta Y}{\Delta D} \tag{1}$$

$$p = \frac{Y_t}{S_t} \tag{2}$$

$$L_t = f\, S_t \tag{3}$$

$$D_t = h\, L_t \tag{4}$$

$$Y_t = L_t^{\,1/2}, \tag{5}$$

where: Y – represents the real gross domestic product; L – labour force; S – population; D – disabled memebers of society; f – the share of population which belongs to labour force; m – the coefficient of marginal productivity of disabled memebers of society; p – the real gross domestic product per capita. Equation (1) determines the marginal productivity of disabled members of society m; equation (2) determines the average productivity, p; equation (3) contains labour force function; equation (4) contains disabled memebers of society function; and equation (5) contains production function.

By substitution one derives:

$$S_{t+1} = \frac{p}{p - mhf} S_t - \frac{m\,h\,p^2}{p - mhf} S_t^2. \tag{6}$$

Further, it is assumed that the current value of population is restricted by its maximal value in its time series. This premise requires a modification of the growth law. Now, the population growth rate depends on the current size of the population, *S*, relative to its maximal size in its time series S^m. We introduce *s* as $s = S/S^m$. Thus s range between 0 and 1. Again we index *s* by *t*, i.e., write s_t to refer to the size at time steps $t = 0, 1, 2, 3, \ldots$ Now growth rate of the population is measured as:

$$s_{t+1} = \frac{p}{p - mhf} s_t - \frac{m\,h\,p^2}{p - m\,h\,f} s_t^2. \tag{7}$$

This model given by equation (7) is called the logistic model. For most choices of *f*, *m*, *h* and *p* there is no explicit solution for (7). Namely, knowing *f*, *m*, *h* and *p* and measuring s_0 would not suffice to predict s_t for any point in time, as was previously possible. This is at the heart of the presence of chaos in deterministic feedback processes. Lorenz (1963) discovered this effect – the lack of predictability in deterministic systems. Sensitive dependence on initial conditions is one of the central ingredients of what is called deterministic chaos.

This kind of difference equation (7) can lead to very interesting dynamic behavior, such as cycles that repeat themselves every two or more periods, and even chaos, in which there is no apparent regularity in the behavior of s_t. This difference equation (7) will posses a chaotic region. Two properties of the chaotic solution are important: firstly, given a starting point s_0 the solution is highly sensitive to variations of the parameters *f*, *m*, *h* and *p*; secondly, given the parameters *f*, *m*, *h* and *p*, the solution is highly sensitive to variations of the initial point s_0. In both cases the two solutions are for the first few periods rather close to each other, but later on they behave in a chaotic manner.

3. Logistic equation

The logistic map is often cited as an example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations. The map was popularized in a seminar 1976 paper by the biologist Robert

May. The logistic model was originally introduced as a demographic model by Pierre François Verhulst.

It is possible to show that iteration process for the logistic equation:

$$z_{t+1} = \pi z_t (1 - z_t), \;\; \pi \in [0,4],\; z_t \in [0,1], \qquad (8)$$

is equivalent to the iteration of growth model (7) when we use the identification:

$$z_t = m\,h\,p\,s_t \text{ and } \pi = \frac{p}{p - m\,h\,f}. \qquad (9)$$

Using (9) and (7) we obtain:

$$z_{t+1} = m\,h\,p\,s_{t+1} = m\,h\,p\left[\frac{p}{p-m\,h\,f}s_t - \frac{m\,h\,p^2}{p-m\,h\,f}s_t^2\right] =$$

$$= \frac{m\,h\,p^2}{p-m\,h\,f}s_t - \frac{m^2\,h^2\,p^3}{p-m\,h\,f}s_t^2.$$

Using (8) and (9) we obtain:

$$z_{t+1} = \pi z_t (1 - z_t) = \left(\frac{p}{p-m\,h\,f}\right) m\,h\,p\,s_t(1 - m\,h\,p\,s_t) =$$

$$= \frac{m\,h\,p^2}{p-m\,h\,f}s_t - \frac{m^2\,h^2\,p^3}{p-m\,h\,f}s_t^2.$$

Thus we have that iterating $s_t + 1 = \frac{p}{p-m\,h\,f}s_t - \frac{m\,h\,p^2}{p-m\,h\,f}s_t^2$ is really the same as iterating $z_{t+1} = \pi z_t (1 - z_t)$ using $z_t = m\,h\,p\,s_t$ and $\pi = \frac{p}{p-m\,h\,f}$. It is important because the dynamic properties of the logistic equation (7) have been widely analyzed (Li and Yorke (1975), May (1976)).

It is obtained that:

- (i) For parameter values $0 < \pi < 1$ all solutions will converge to $z = 0$;
- (ii) For $1 < \pi < 3{,}57$ there exist fixed points the number of which depends on π;

- (iii) For $1 < \pi < 2$ all solutions monotnically increase to $z = (\pi - 1)/\pi$;
- (iv) For $2 < \pi < 3$ fluctuations will converge to $z = (\pi - 1)/\pi$;
- (v) For $3 < \pi < 4$ all solutions will continously fluctuate;
- (vi) For $3{,}57 < \pi < 4$ the solution become "chaotic" wihch means that there exist totally aperiodic solution or periodic solutions with a very large, complicated period. This means that the path of z_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatever.

4. Empirical evidence

The main aim of this paper is to analyze the population growth stability in the EU-27 countries in the period 1998-2009. by using the presented non-linear, logistic growth model (10):

$$s_{t+1} = \alpha\, s_t - \beta\, s_t^2 \tag{10}$$

where:

$$s - \text{represents population},\ \alpha = \frac{p}{p - m\,h\,f},\ \ \beta = \frac{m\,h\,p^2}{p - m\,h\,f}.$$

Firstly, we transform data on population [Source: Eurostat Yearbook 2006-07] from 0 to 1, according to our supposition that actual value of population, S, is restricted by its highest value in the time-series, S^m. Further, we obtain time-series of $s = S/S^m$. Now, we estimate the model (10). The results are presented: (Source: Eurostat Yearbook 2006-07, European Commission):

- EU-27 1998-2009:

R = .99808 Variance explained: 99.616%

	α	β
Estimate	.91761	–.08796
Std.Err.	.02248	.02302
$t(9)$	40.82530	–3.82167
p – level	.00000	.00408

5. Conclusions

This paper suggests conclusion for the use of the chaotic population growth model in predicting the fluctuations of the population. The model (7) has to rely on specified parameters f, m, p, h and initial value of the population, s_0. But even slight deviations from the values of parameters f, m, p, h, and initial value of the population, s_0, show the difficulty of predicting a long-term population behaviour.

A key hypothesis of this work is based on the idea that the coefficient $\pi = \frac{p}{p - m\,h\,f}$ plays a crucial role in explaining local population stability, where f represents the share of population which belongs to labour force; m – the coefficient of marginal productivity of disabled members of society; p – the real gross domestic product per capita; f – the share of population which belongs to labour force. An estimated values of the coefficient $\pi = \frac{p}{p - m\,h\,f} = 0.91761$ in the EU-27 countries is smaller than 1 in the observed period. This result confirms stable but declining population growth in the EU-27 countries in the observed period.

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