DESCRIPTION AND ANALYSIS OF FUZZY INFORMATION

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1. Overview

Results of measurements and observations are important information. This information is even more than only numbers or numerical vectors. One of the problems are that a lot of information is imprecise or fuzzy.

Even in the 15th century Nikolaus von Kues (Cusanus) stated the "basically non-avoidable impreciseness of measurements". The physician Robert von Mayer (1814-1878) said that numbers are the basis of an "exact science". Galileo Galilei asked for "measure all which can be measured and to make things measurable which cannot already be measured".

Very often a part of uncertainty is included in information. In most of these cases it is caused by a lingual uncertainty which we all already found have in our life. E.g. we say that a management of a company is "experienced", "already experienced" or "not experienced". This is a typical example of basis information of some rating systems which are used in banks.

The mathematical description of such information – using exact real numbers – is not possible. Even "obvious exact" information is in fact fuzzy if we have a closer look on it.

Types of uncertainty or fuzziness which is included (very often hidden) in information are

- Randomness
- Errors of measurements
- Uncertainty in used models

Of course this uncertainty or fuzziness can be caused by combinations of the types which are listed above.

It is within the "nature of a complex system" that some information is "fuzzy". So it is very important to be able to describe it in mathematical models in order to be able to describe complex systems.

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2. Description of fuzzy numbers

If we try to analyze information or observations we always use numbers. As mentioned before most of the available information or observations are not exact but fuzzy. Based on the idea of Karl Menger (1902-1985) a set can be characterized by an indicator function. So if we have a set **A** which is subset of **M** the indicator function is defined by:

$$I_A : M \to \{0,1\},\ A \subset M$$

$$I_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases} \qquad \forall\, x \in M$$

1951 Menger published his idea of fuzzy sets. Now we have to consider how fuzzy numbers can be described.

Definition 1:

A fuzzy subset A^* of a set M is described by a so called membership function $\mu_{A^*}(.)$ which is:

$$\mu_{A^*} : M \to [0,1].$$

A fuzzy set A^* is called normalised if:

$$\exists\, x \in M : \mu_{A^*}(x) = 1.$$

Similar to this we can describe a fuzzy number x^* by a characterizing function.

Definition 2:

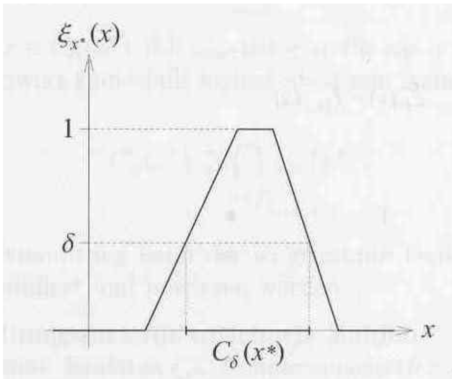
A real function $\xi_{x^*}(.)$ is called a characterizing function of a fuzzy number x^*, if following conditions are fulfilled:

(1) $\xi_{x^*} : R \to [0,1]$

(2) $\forall\, \delta \in (0,1]$ the δ-cut $C_\delta(x^*) := \{x \in R : \xi_{x^*}(x) \geq \delta\}$ is a finite, non-empty and closed interval. So $C_\delta(x^*)$ is always a compact interval.

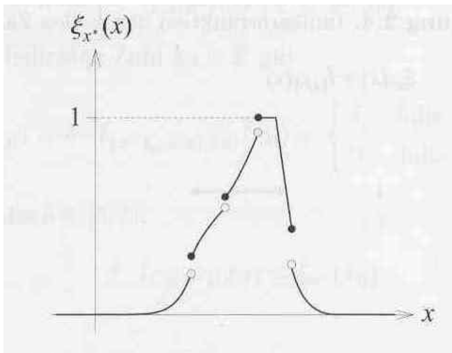
Examples for characterizing functions are represented in the figure following

- Characterizing function is a trapezoid

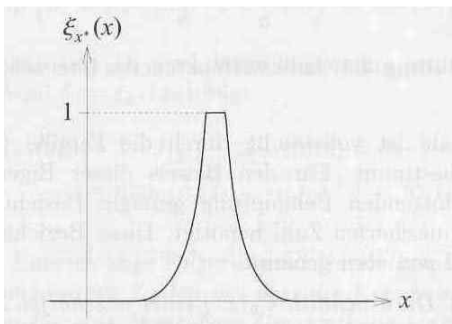


This figure shows a δ-Cut $C_\delta(x^*)$ also.

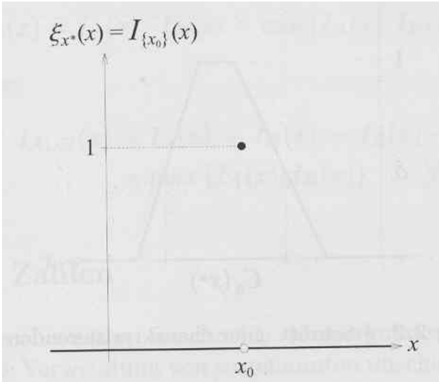
- A non-continuous characterizing function



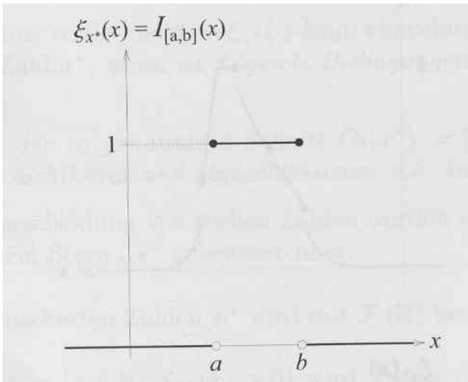
- A continuous characterizing function



- An indicator function which describes a real number



- An indicator function for the interval [a, b] with $a, b \in R$



If there is n-dimensional information we describe it by fuzzy vectors. Fuzzy vectors are described by vector characterizing functions.

Definition 3:

A function $\xi_{x^*}(.,...,.)$ of n real variables is called a vector-characterizing function of a n-dimensional fuzzy vector x^* if following conditions are fulfilled:

(1) $\xi_{x^*} : R^n \to [0,1]$

(2) $\forall\, \delta \in (0,1]$ the δ-cut $C_\delta(x^*) := \{x \in R^n : \xi_{x^*}(x) \geq \delta\}$ is a imply connected and compact subset of R^n.

Even very interesting is to consider functions of fuzzy numbers. For this we do the following definition.

Definition 4:

The supremum of a function $f\colon R^n \to R^k$ over an empty set $\varnothing \subseteq R^n$ is defined as zero:

$$\sup\{f(x): x \in \varnothing\} := 0$$

If we have to do some statistical methods of observations which are fuzzy we have to do it very often on functions of these observations. So we have to consider an enhancement of real functions even to be able to operate with functions of fuzzy numbers.

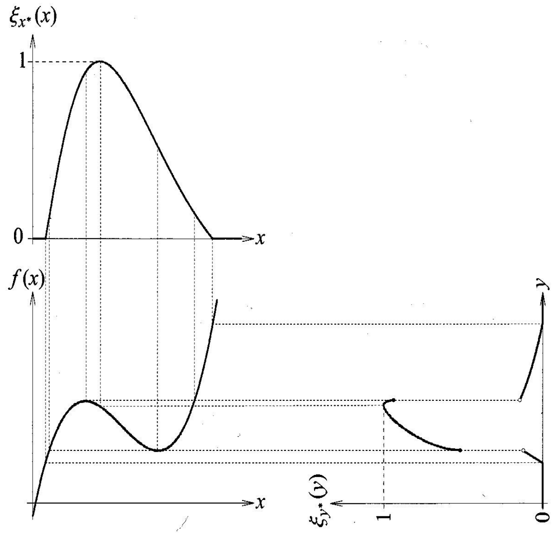
Principle of enhancement:

For a real function $f\colon R^n \to R^k$ the characterizing function $\xi_{y^*}(.,...,.)$ of a fuzzy vector $y^* = f(x^*)$ is defined by:

$$\xi_{y^*}(y) := \begin{cases} \sup\{\xi_{x^*}(x): f(x) = y\} & \text{if } \exists\, x \in R^n:\ f(x) = y \\ 0 & \text{if } \neg\exists\, x \in R^n:\ f(x) = y \end{cases}$$

for all $y \in R^k$ and assumed that $\xi_{x^*}(.,...,.)$ is the characterizing function of x^*.

In the following picture an application of this principle is shown.



3. Mathematical operations on fuzzy numbers

Now we have to consider how operations on fuzzy numbers have to be defined. Each result of such an operation is a fuzzy number. So it is obvious that even the result of an operation on fuzzy number is identified by a characterizing function.

So the fuzzy operation $x^* = x_1^* \oplus x_2^*$ can be seen as function $f(x_1, x_2) = x_1 + x_2$. According to the "principle of enhancement" the characterizing function of the sum of two fuzzy numbers can be calculated as:

$$\xi_{x^*}(x) = \xi_{x_1^* \oplus x_2^*}(x) = \sup\{\min(\xi_{x_1^*}(x), \xi_{x_2^*}(x)) : (x_1, x_2) \in R^2 \text{ and } x_1 + x_2 = x\}$$

$$= \sup\{\min(\xi_{x_1^*}(y), \xi_{x_2^*}(x - y)) : y \in R\}.$$

For the multiplication of fuzzy numbers $x^* = x_1^* \otimes x_2^*$ which is like $f(x_1, x_2) = x_1 \cdot x_2$ the product of two fuzzy numbers is given by:

$$\xi_{x^*}(x) = \xi_{x_1^* \otimes x_2^*}(x) = \sup\{\min(\xi_{x_1^*}(x), \xi_{x_2^*}(x)) : (x_1, x_2) \in R^2 \text{ and } x_1 \cdot x_2 = x\}.$$

A special case is $y^* = \lambda \otimes x^*$. Here the characterizing function is:

$$\xi_{y^*}(y) = \xi_{\lambda \otimes y^*}(y) = \sup\{\xi_{x^*}(x) : x \in R \text{ and } \lambda \cdot x = y\}$$

$$= \begin{cases} \xi_{x^*}(\lambda^{-1} \cdot y^*) & \text{if } \lambda \neq 0 \\ I_{\{0\}}(y) & \text{if } \lambda = 0 \end{cases}.$$

If we now consider the MIN-function of a fuzzy number we have to define it by using δ-cuts of the characterizing function of this fuzzy number. So if $C_{i,\delta}$ is the δ-cut of x_i^* with:

$$C_{i,\delta} = [a_{i,\delta}, b_{i,\delta}], \ \delta \in (0,1], \ i = 1,\ldots,n$$

the δ-cuts of $\min[x_1^*,\ldots,x_n^*]$ are defined as:

$$C_\delta = \left[\min_{i=1,\ldots,n} a_{i,\delta}, \min_{i=1,\ldots,n} b_{i,\delta}\right] \qquad \forall\ \delta \in (0,1].$$

The characterizing function $\xi(.)$ of the fuzzy $\min[x_1^*,\ldots,x_n^*]$ is given by:

$$\xi(x) = \max_{\delta \in (0,1]} \delta \cdot I_{C_\delta}(x) \qquad \forall x \in R.$$

In the same way we get the MAX-function. The δ-cut of the fuzzy MAX-function is defined as:

$$C_\delta = \left[\max_{i=1,\ldots,n} a_{i,\delta}, \max_{i=1,\ldots,n} b_{i,\delta}\right] \qquad \forall \delta \in (0,1].$$

The characterizing function $\xi(.)$ of the fuzzy $\max[x_1^*,...,x_n^*]$ is given by:

$$\xi(x) = \max_{\delta \in (0,1]} \delta \cdot I_{C_\delta}(x) \qquad \forall x \in R.$$

4. Statistical Analysis of fuzzy information

If we now try to describe fuzzy observation in a statistical way, we will get for the:

– *upper limit of the fuzzy relative frequency*

$$\bar{h}_{n,\delta}(K_i) = \frac{\#\{x_j^* : C_\delta(x_j^*) \cap K_i \neq \emptyset\}}{n}$$

– *lower limit of the fuzzy relative frequency*

$$\underline{h}_{n,\delta}(K_i) = \frac{\#\{x_j^* : C_\delta(x_j^*) \subseteq K_i\}}{n}$$

where:

#... is the number of elements in the set

and:

K_i… are disjunctive classes.

The δ-cut of the *fuzzy relative frequency* $h_n^*(K_i)$ can be calculated as follows:

$$C_\delta(h_n^*(K_i)) = [\underline{h}_{n,\delta}(K_i), \bar{h}_{n,\delta}(K_i)].$$

The *empirical distribution function* of fuzzy observations is defined by:

$$\hat{F}_n^*(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} \frac{\int_{-\infty}^{x} \xi_{x_i^*}(t)\mathrm{d}t}{\int_{-\infty}^{\infty} \xi_{x_i^*}(t)\mathrm{d}t} \qquad \forall x \in R$$

where:

$x_1^*, ..., x_n^*$ are fuzzy observations

and:

$\xi_{x_1^*}(.), ..., \xi_{x_n^*}(.)$ are the characterizing functions of the fuzzy observations and

we assume that:

$$\int_{-\infty}^{\infty} \xi_{x_i^*}(t)\mathrm{d}t \neq 0 \qquad \text{for } i = 1, ..., n .$$

If we have a mixture of normal and fuzzy observations we get:

$$\hat{F}_n^*(x) = \frac{1}{n} \cdot \sum_{i=1}^{k} I_{(-\infty,\, x]}(y_i) + \frac{1}{n} \cdot \sum_{i=1}^{l} \frac{\int_{-\infty}^{x} \xi_{x_i^*}(t)\mathrm{d}t}{\int_{-\infty}^{\infty} \xi_{x_i^*}(t)\mathrm{d}t} \qquad \forall\, x \in R$$

where:

$y_1, ..., y_k$ are normal observations

and

$x_1^*, ..., x_1^*$ are fuzzy observations

and

$k + l = n$ observations in our sample

and

$\xi_{x_1^*}(.), ..., \xi_{x_l^*}(.)$ are the characterizing functions of the fuzzy observations

and

we assume that:

$$\int_{-\infty}^{\infty} \xi_{x_i^*}(t)\mathrm{d}t \neq 0 \qquad \text{for } i = 1, ..., 1 .$$

Examples:

The following examples show that fuzzy observations are in almost every topic where we believe that observations are exact.

Impossibility of exact data definition

The loss caused by black economy cannot be estimated to due non available information. Even the definition of “black economy” is a very fuzzy one.

Different information

Different vendors of the same product will sell with different prices. So we cannot answer the question of THE PRICE of a product. THE PRICE of a product can be described by the min and max value which we already know.

Limited precision of analyzers

All analyzers which are used to measure continuous values have a very limited scale and even limited precision. Digital analyzers have a finite number of decimal digits on their displays.

Missing objective scale

The value of a building or a valuable is influenced by attributes of the person who gives estimation for this. Even if we evaluate the quality of a management we are influenced by our own opinion.

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